From Trees to Closed Loops: Inventory Management in Treewidth-Bounded Supply Chain Networks

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Abstract. We provide a novel analysis of the *Guaranteed Service Model* (GSM), one of the most widely applied models for multi-echelon inventory management. In particular, we develop a procedure to solve the GSM in time exponential in the *treewidth* of the underlying supply chain network graph, but linear in the number of nodes n of the graph. The treewidth of a graph describes the graph's similarity to a tree—it is one for a tree graph, and $n - 1$ for a fully connected graph. Our procedure is based on solving a linear program, so it can easily be implemented with standard solvers. This allows solving the GSM for large and highly complex supply chain networks, as long as the underlying network's treewidth is relatively small—which, recent literature has identified, is the case for supply chain networks encountered in practice.

We then extend the GSM and our procedure to closed-loop supply chain networks, that is, networks with reverse flows. We show that individual reverse flows do not significantly affect the solution time. This opens the door to analyzing inventory management in the circular economy and evaluating the viability of different circular business models.

Key words: supply chain management; inventory management; algorithms; computational complexity; circular economy

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1. Introduction

Supply chain networks are evolving rapidly, becoming increasingly global, complex, and circular. The emergence of circular business models, supporting servitization, recycling, and product returns, has introduced "closed loops" into these networks. This evolution presents new challenges in inventory management, particularly for firms aiming to balance sustainability with economic growth (Calmon and Graves 2017, Agrawal et al. 2019). Traditional inventory models, developed for simpler, non-circular supply chains, are often inadequate for these complex, closed-loop systems.

A prime example of this limitation is the Guaranteed Service Model (GSM), a widely used approach for optimizing safety stock placement in supply chains (Eruguz et al. 2016, Schoenmeyr and Graves 2022).[1] Despite its popularity, the GSM, an NP-hard problem (Lesnaia 2004), has

[1] The GSM has been used by many major corporations, including Hewlett-Packard (Billington et al. 2004), Proctor & Gamble (Farasyn et al. 2011), Intel (Manary et al. 2019), and at least 19 others (2019 presentation based on an early draft of Schoenmeyr and Graves 2022). Three applications of the GSM were finalists for the *Franz Edelman Award* (Schoenmeyr and Graves 2022).

predominantly been considered for tree structures or networks with limited complexity, such as extensions of trees, where individual nodes are substituted by "clusters of commonality". Moreover, traditional solution approaches typically rely on custom dynamic programming algorithms, making their extension to more general, potentially closed-loop network structures challenging.

This paper addresses these limitations by generalizing the GSM to more complex, potentially closed-loop supply chains with bounded treewidth—a graph-theoretic measure quantifying how "tree-like" a graph is. We note that there is recent evidence that real-world supply chains have low treewidth (Blaettchen et al. 2024). To obtain this generalization, we introduce a novel theoretical framework for addressing optimization problems in supply chain networks, leveraging recent advances in Integer Programming. By utilizing the *intersection* graph induced by the constraints of the integer program, we reformulate the GSM as a linear program (LP).

Our approach offers two key advantages. First, we establish an explicit link between the treewidth of a supply chain network and the complexity of solving the GSM within that network. We prove that the complexity of solving the GSM is exponential only in the supply chain network's treewidth —i.e. it is *fixed-parameter tractable (FPT)* in the treewidth —irrespective of the presence of reverse flows. This provides a theoretical foundation for understanding the computational complexity of the GSM in relation to network structure. Second, our framework allows for the reformulation of the GSM in any treewidth-bounded network as an LP, solvable using standard optimization software, thus eliminating the need for tailored algorithms.

We present several new results about the GSM. Notably, when the underlying supply chain network is a tree, we prove that the GSM can be reformulated as an LP for any increasing (not necessarily convex or concave) objective function—the first such linear reformulation of the GSM with this level of generality. For non-tree networks, including those with reverse flows, we introduce an LP reformulation of the GSM with a number of variables and constraints exponential only in the underlying treewidth of the supply chain network.

The ability to solve the GSM for general networks opens up the possibility of examining the effects of introducing reverse flows on inventory placement within the network. In particular, it allows for the study of different types of reverse flows (both in intensity and location), enabled by diverse circular business models.

This research contributes to both the theoretical understanding and practical implementation of inventory management in complex supply chains. For researchers, it introduces a new framework for approaching complex supply chain optimization problems. For practitioners, it offers a practical approach to safety-stock placement in general supply chain networks without requiring custom algorithm development. Rather, practitioners only need access to an LP solver, the most powerful of which are increasingly open-source (e.g., the PDLP solver presented in Applegate et al. 2021).

In summary, by extending the GSM to complex, closed-loop supply chains, we develop a new framework for approaching the theoretical characterization, optimization, and practical implementation of supply chain management problems. This new bridge between recent integer optimization research and modern supply chain problems can help develop new computational and strategic approaches that result in more sustainable operations.

The remainder of this paper is organized as follows. In Section 2 we review the GSM. Then, in Section 3 we reformulate the GSM for supply chain networks that are trees as an LP. In Section 4, we extend the GSM to general networks. Section 5, extends the GSM to closed-loop networks.

2. The Guaranteed Service Model

We next describe here the standard assumptions of the Guaranteed Service Model (see, e.g., Graves and Willems 2000). We consider a directed, acyclic supply chain network $G = (V, E)$, with $n = |V|$ nodes, and edges representing input-output relationship. That is, if $(i, j) \in E$, then each unit produced by j requires one unit of input from i.[2] We let L be the set of nodes in the graph for which there are no outgoing arcs, also referred to as *demand nodes*.

Each node operates a periodic base-stock-policy, with a common review period across nodes, that is normalized to one. Whenever a nodes observes demand, it will immediately proceed to processing the quantity demanded. In particular, when a demand node j observes external demands D_j^t for some period $t \in \mathbb{N}$, this immediately implies that a corresponding demand is observed at all upstream nodes. Note that this does not require any explicit information sharing across the network. Rather, all nodes immediately trigger orders from their immediate suppliers due to use of a base-stock-policy. While demands across demand nodes may be arbitrarily correlated, we assume that demands in each period t independently follow the same distributions.

In order to fulfill a certain demand, node j requires inputs from its predecessors. Moreover, it has a (deterministic) processing time T_j, independent of the quantity processed. To fulfill demands quickly, the node will stock a certain quantity (or *base stock*) B_j. Rather than directly deciding on the base stock, the GSM converts this decision into deciding the units of times for which inventory should cover demand. This enables a simpler formulation of the relevant constraints.

[2] This is an assumption made for simpler exposition. Besides the precise objective formulation, the problem structure is unaffected if production of one unit at j requires $\phi \in \mathbb{N}_+$ inputs from i.

First, node j guarantees an *outbound service time* S_j to its downstream neighbors (or outside customers, in the case of nodes $j \in F$). This reflects the number of periods within which the node guarantees that all demand will be met, as long as that demand quantity is within predetermined bounds. For demand nodes, there may be an upper bound on the service time based on customer requirements, i.e. $S_j \leq s_j$ for some $s_j \in \mathbb{N}$.

Second, node j will have a guaranteed *inbound service time* SI_j, that is, the number of times it will take to receive all relevant inputs from its predecessors. While there may be flexibility in setting this time, the inbound service time cannot be lower than the outbound service time of any predecessor. That is, $SI_j \geq \max\{S_i : (i,j) \in E\}$.

Using the service times, we are ready to compute the expected inventory at node i and, thus, its costs. Say that node j guarantees outbound service time S_j and inbound service time SI_j. Here, we can assume $S_i \leq SI_i + T_i$ without loss of generality: if the outbound service time were larger than the time it takes to acquire input and process it, the inventory is always zero.

In each period, node j observes stochastic demand D_j^t. This could be directly observed for demand nodes, or it could be a combination of different stochastic demands, based on how these demands are propagated through the network of suppliers. In either case, this demand follows a time-invariant mean μ_j. Moreover, the outbound service time guarantee by node j covers a deterministic upper bound $\overline{D}_j(\tau)$. For example, this could correspond to the 95-th percentile of demand across τ periods. A key assumption of the GSM is that any demand not covered by this bound is dealt with through measures outside the model.

Let $I_j(t)$ be the inventory of node j at time t and assume, without loss of generality, that $I_j(0) = B_j$. Then, at time t, the inventory is (cf. Klosterhalfen et al. 2014)

$$I_j(t) = B_j - \sum_{m=1}^{t-S_j} D_j^m + \sum_{m=1}^{t-T_j-SI_j} D_j^m.$$

Here, the first sum represents the demands that need to be covered (any demand up to S_j periods before the current period). The second sum represents the output produced by the node (guaranteed to be equal to demand up to $SI_j + T_j$ periods before the current period). To ensure $I_j(t) \geq 0$ as long as demand does not exceed the upper bound, we require that

$$B_j \geq \overline{D}_j(SI_j + T_j - S_j).$$

If we assume that the base stock B_j is exactly at the lower bound (so that each node fulfills its service time promise but does not hold inventory in excess), we can then compute the long-term average inventory held by node j:

$$\begin{aligned}\lim_{t\to\infty}\frac{\sum_{m=1}^{t} I_j(m)}{t} &= \overline{D}_j(SI_j+T_j-S_j) - \frac{\sum_{m=1}^{t-S_j} D_j^m}{t} + \frac{\sum_{m=1}^{t-T_j-SI_j} D_j^m}{t}\\ &= \overline{D}_j(SI_j+T_j-S_j) - \mu_j(SI_j+T_j-S_j).\end{aligned}$$

This average inventory, or safety stock, is multiplied by a node-specific inventory cost h_j and summed across all nodes $j \in V$ to arrive at the total expected inventory cost. Then, noting that the work-in-progress inventory is independent of the service times and base stocks, we can formulate the GSM as the following optimization problem:

$$\begin{aligned}\min_{S_j,SI_j}\quad & \sum_{j=1}^{n} f_j(SI_j - S_j)\\ \text{s.t.}\quad & S_j - SI_j \le T_j,\ j=1,\dots,n\\ & SI_j - S_i \ge 0,\ (i,j)\in E\\ & S_j \le s_j,\ j\in L\\ & S_j, SI_j \ge 0 \text{ and integer for } j=1,\dots,n.\end{aligned}\tag{1}$$

Here, we have replaced $h_j\left[\overline{D}_j(SI_j+T_j-S_j)-\mu_j(SI_j+T_j-S_j)\right]$ with a more generic non-decreasing function $f:\mathbb{R}\to\mathbb{R}$. While f is usually assumed to be concave in the literature, we expressly do not make this assumption.

Problem (1) does not have a bounded feasible set. To remedy this, consider the set $paths(G,j)$ of paths in G ending in node j. We define the maximum replenishment time $M_j = \max_{p\in paths(G,j)}\sum_{i\in p} T_i$, that is, the maximum time it takes to produce all the inputs for node j and the product at node j itself.

Under the assumption that $f_j(\cdot)$ is nondecreasing, it can be shown that $SI_j \le M_j - T_j$ for any $j=1,\dots,n$ (see Graves and Willems 2000). As $S_j \le SI_j + T_j$, it follows that $S_j \le M_j$ for any $j=1,\dots,n$. Because $M_j \le M := \max_{p\in paths(G)}\sum_{i\in p} T_i$, one can add the constraints $SI_j, S_j \le M$ for any $j=1,\dots,n$ to (1) without changing the optimal solution, nor the optimal value of the problem. Note that $M \le \sum_{j=1}^{n} T_j$. Furthermore, to simplify notation moving forward, we take:

$$s_j = +\infty \text{ for } j\notin L \text{ and } \Gamma_j = \min\{M, s_j\},\ \text{for } j=1,\dots,n.$$

Note that $\Gamma_j \leq M$ for $j = 1, \ldots, n$. Thus, in the remainder, our goal will be to solve the optimization problem

$$\min_{S_j, SI_j} \quad \sum_{j=1}^{n} f_j(SI_j - S_j) \tag{2}$$

$$\text{s.t.} \quad S_j - SI_j \leq T_j, \; j = 1, \ldots, n, \tag{$2a_j$}$$

$$SI_j - S_i \geq 0, \; (i, j) \in E, \tag{$2b_{ij}$}$$

$$0 \leq S_j \leq \Gamma_j \text{ and } S_j \text{ integer for } \; j = 1, \ldots, n, \tag{$2c_j$}$$

$$0 \leq SI_j \leq M \text{ and } SI_j \text{ integer for } \; j = 1, \ldots, n, \tag{$2d_j$}$$

which is equivalent to the original GSM.

3. Solving the GSM on a tree via linear programming

In this section, we assume that G is a tree. It was shown in Graves and Willems (2000) that solving the GSM on a tree can be done via dynamic programming in time $O(nM^2)$ where, as mentioned above, n is the number of nodes in the graph and $M = \max_{p \in paths(G)} \sum_{i \in p} T_i$. We propose here to use a network flow formulation which enables us to write the GSM over a tree as a linear program with $O(nM^2)$ variables and constraints. The advantage of using a linear programming formulation over a dynamic programming formulation is twofold: (i) solving a linear program does not require a custom-made solver as is needed for a dynamic program but can be solved via a general purpose solver; (ii) adding constraints or changing the objective of the model can be done very straightforwardly within the linear program.

The case of G being a tree is a subcase of the case considered in Section 4, where we assume that G has bounded treewidth ω (the definition of treewidth is given in Definition 2). Indeed, a tree has treewidth $\omega = 1$. This section serves as an important primer for Section 4: we are able to introduce relevant concepts and a proof outline for our main result, without cumbersome notation. The proof of the main result of Section 4 is essentially a generalization of what is done here. Furthermore, it may be of interest in some settings to have the explicit formulation of the GSM for the tree case.

We start by stating the main theorem (Theorem 1) in Section 3.1. In Section 3.3, we present an overview of the key proof steps, with the proof itself in Appendix B. These steps require some concepts, such as tree decompositions of graphs, which we introduce in Section 3.2.

3.1. Main theorem of the section

The main theorem of this section gives an explicit formulation of the GSM as a linear program in $O(nM^2)$ variables and $O(nM^2)$ constraints. It requires the introduction of the set:

$$\mathcal{S}_{ij} = \{(k_i, \ell_j) \in \{0, \dots, \Gamma_i\} \times \{0, \dots, M\} \mid k_i > \ell_j\}, \text{ for } (i, j) \in E \tag{3}$$

and the definition of the standard simplex:

$$\Delta_n = \{(x_0, x_1, \dots, x_n) \mid \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0, i = 0, \dots, n\}.$$

Note that $\Delta_n \subseteq \mathbb{R}^{n+1}$ is an n-dimensional set as indicated by its subscript. In the reminder, to keep notation to a minimum, we will sometimes drop this subscript: it simply corresponds to the size of the set which belongs to the simplex, minus one.

THEOREM 1. *When $G = (V, E)$ is a tree, the guaranteed service model given in (2) can be formulated as the following linear program in $O(nM^2)$ variables and $O(nM^2)$ constraints:*

$$\min \quad \sum_{j=1}^{n} \sum_{\ell_j=0}^{M} \sum_{k_j=0}^{\Gamma_j} f_j(\ell_j - k_j) r^j_{k_j \ell_j} \tag{4}$$

$$\text{s.t.} \quad r^j_{k_j \ell_j} \leq \left\lfloor \frac{T_j}{k_j - \ell_j} \right\rfloor, \ \forall k_j = 0, \dots, \Gamma_j,\ \ell_j = 0, \dots, M,\ k_j > \ell_j, j = 1, \dots, n \tag{$4a_j$}$$

$$s^{ij}_{k_i \ell_j} = 0,\ \forall (k_i, \ell_j) \in \mathcal{S}_{ij},\ \forall (i, j) \in E \tag{$4b_{ij}$}$$

$$\sum_{k_j=0}^{\Gamma_j} r^j_{k_j \ell_j} = \sum_{k_i=0}^{\Gamma_i} s^{ij}_{k_i \ell_j},\ \forall \ell_j = 0, \dots, M,\ \forall j \in \{1, \dots, n\} \text{ s.t. } (i, j) \in E, \tag{$4c_j$}$$

$$\sum_{\ell_i=0}^{M} r^i_{k_i \ell_i} = \sum_{\ell_j=0}^{M} s^{ij}_{k_i \ell_j},\ \forall k_i = 0, \dots, \Gamma_i,\ \forall i \in \{1, \dots, n\} \text{ s.t. } (i, j) \in E, \tag{$4d_i$}$$

$$\{r^j_{k_j \ell_j}\}_{k_j, \ell_j} \in \Delta,\ \forall j = 1, \dots, n, \tag{$4e_j$}$$

$$\{s^{ij}_{k_i \ell_j}\}_{k_i, \ell_j} \in \Delta,\ \forall (i, j) \in E, \tag{$4f_{ij}$}$$

with decision variables $\{r^j_{k_j \ell_j}\}_{k_j=0,\dots,\Gamma_j,\ \ell_j=0,\dots,M,\ j=1,\dots,n}$ *and* $\{s^{ij}_{k_i \ell_j}\}_{k_i=0,\dots,\Gamma_i,\ \ell_j=0,\dots,M,\ (i,j)\in E}$.

The proof of the theorem is deferred to Appendix B, but we give, in Section 3.3, an intuitive and high-level explanation of the key steps of the proof. To do this, however, we need to introduce some concepts, which we illustrate in Figure 1.

3.2. Some important concepts

We first introduce the notion of an *intersection graph* for an optimization problem. The intersection graph is a concise way of visualizing how variables interact within the constraints and objective function of an optimization problem. We introduce it here specifically for (2), though it can be defined more broadly. Recall that a function $f:\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_p}\to\mathbb{R}$ is said to be separable if it can be written $f(x_1,\ldots,x_p)=g_1(x_1)+\ldots+g_p(x_p)$ for some $x_i\in\mathbb{R}^{n_i}$ and functions $g_i:\mathbb{R}^{n_i}\to\mathbb{R}$ for $i=1,\ldots,p$. Note that the objective function of (2) is separable.

Definition 1. The intersection graph G' for (2) is an undirected graph where each node represents a decision variable S_i or SI_i for $i=1,\ldots,n$, and there is an edge between two nodes if they appear in the same constraint or in the same term of the separable objective.

Given a supply chain network G which is a directed graph, it is quite straightforward to build the intersection graph G' of (2), as explained below.

Proposition 1. *Given a directed graph G, the intersection graph G' of (2) is obtained by replacing each node $i\in G$ by a subgraph of two nodes SI_i and S_i linked by an edge, with all ingoing edges to i rerouted to SI_i and any outgoing edges from i being outgoing from S_i.*

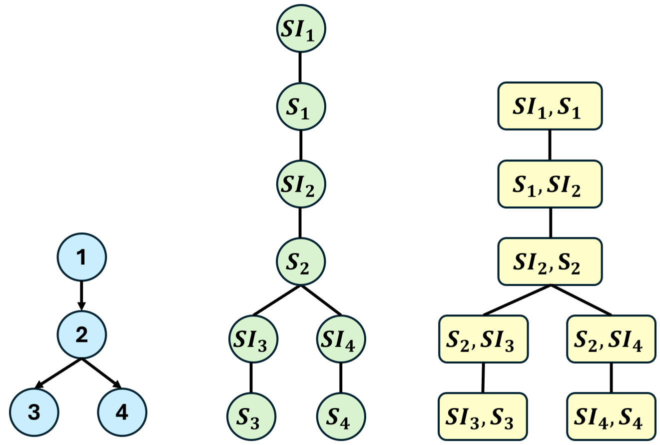
As an illustration, consider the supply chain network G given in Figure 1a. Its intersection graph G' is given in Figure 1b and is built as indicated in Proposition 1. Note that this result is more generally applicable than trees.

Proof. We argue that the graph obtained is the intersection graph of G. By replacing any node i by two nodes S_i and SI_i, it is clear that all decision variables appear as nodes in the intersection graph. Furthermore, three types of edges should feature in the intersection graph: one is due to the constraints ($2a_j$), another one is due to the constraints ($2b_{ij}$), and the last is due to the objective function of (2) (constraints ($2c_j$) and ($2d_j$) do not lead to edges in the graph). Constraints ($2a_j$) and the structure of the objective function of (2) lead to edges between S_j and SI_j for all $j=1,\ldots,n$. These are taken care of due to the replacement of i by the subgraph $(SI_i)-(S_i)$. Constraints ($2b_{ij}$) lead to edges between S_i and SI_j for any $(i,j)\in E$. These are obtained by rerouting the ingoing/outgoing edges of i. □

We now introduce the notion of a tree decomposition which can be defined for any graph:

Definition 2 (Bodlaender 1994). Let $G=(N,E)$ be a graph. A *tree decomposition* of G is a pair $\mathcal{T}=(T,\{X_z\}_{z\in T})$ with tree T and bags X_z for each node $z\in T$ such that:

(a) $\cup_{z\in T}X_z=N$



(a) G (b) G' (c) $\mathcal{T}'$

Figure 1 Important graphs in the proof of Theorem 1: the original supply chain graph G, the intersection graph G', and its tree decomposition $\mathcal{T}'$.

(b) If $(i, j) \in E$ then there must be a bag X_z with $i, j \in X_z$.

(c) If a node $i \in N$ appears in two distinct bags X_x and X_y, then it appears in all bags X_z such that z is on the (unique) path betweeen x and y in T.

The tree decomposition's *width* is $\max_{z \in T} |X_z| - 1$. The treewidth $tw(G)$ is simply the minimum width over all tree decompositions of G.

A tree decomposition $\mathcal{T}'$ of G' is given in Figure 1c. One can check that the conditions above are satisfied. Note that $tw(G') = 1$. Furthermore, the bags $\{X_z\}_{z \in T}$ contain decision variables here.

Finally, we require the notion of a *join* of two polytopes that share a common set of variables. This definition of a join is specific to this paper.

Definition 3. For $i = 1, 2$, let Q_i be a polytope in variables $(x_i, y) \in \mathbb{R}^{n_i} \times \mathbb{R}^p$. Assume that the projections of Q_1, Q_2 in the space of y form a common set C. Then, their *join* $Q_1 \wedge Q_2$ is given by

$$Q_1 \wedge Q_2 = \{(x_1, x_2, y) \mid (x_1, y) \in Q_1, (x_2, y) \in Q_2\}.$$

3.3. High-level overview of the proof of Theorem 1

The proof of Theorem 1 relies on an equivalent formulation of Problem 4 which we give here:

$$\min_{\{\lambda^j_{\ell_j}\}, \{\gamma^j_{k_j}\}, \{r^j_{k_j \ell_j}\}, \{s^{ij}_{k_i \ell_j}\}} \quad \sum_{j=1}^{n} \sum_{\ell_j=0}^{M} \sum_{k_j=0}^{\Gamma_j} f_j(\ell_j - k_j) r^j_{k_j \ell_j} \tag{5}$$

$$\text{s.t.} \quad r^j_{k_j \ell_j} \le \left\lfloor \frac{T_j}{k_j - \ell_j} \right\rfloor, \ \forall k_j = 0, \ldots, \Gamma_j, \ \ell_j = 0, \ldots, M, \ k_j > \ell_j, j = 1, \ldots, n \tag{$5a_j$}$$

$$s^{ij}_{k_i \ell_j} = 0, \ \forall (k_i, \ell_j) \in \mathcal{S}_{ij}, \ \forall (i, j) \in E \tag{$5b_{ij}$}$$

$$\lambda^j_{\ell_j} = \sum_{k_i=0}^{\Gamma_i} s^{ij}_{k_i\ell_j},\ \forall \ell_j = 0,\dots,M,\ \forall j \in \{1,\dots,n\} \text{ s.t. } (i,j)\in E, \tag{5c_j}$$

$$\lambda^j_{\ell_j} = \sum_{k_j=0}^{\Gamma_j} r^j_{k_j\ell_j},\ \forall \ell_j = 0,\dots,M,\ \forall j = 1,\dots,n, \tag{5d_j}$$

$$\gamma^i_{k_i} = \sum_{\ell_j=0}^{M} s^{ij}_{k_i\ell_j},\ \forall k_i = 0,\dots,\Gamma_i,\ \forall i \in \{1,\dots,n\} \text{ s.t. } (i,j)\in E, \tag{5e_i}$$

$$\gamma^j_{k_j} = \sum_{\ell_j=0}^{M} r^j_{k_j\ell_j},\ \forall k_j = 0,\dots,\Gamma_j,\ \forall j \in \{1,\dots,n\}, \tag{5f_j}$$

$$\{r^j_{k_j\ell_j}\}_{k_j,\ell_j} \in \Delta,\ \forall j = 1,\dots,n, \tag{5g_j}$$

$$\{s^{ij}_{k_i\ell_j}\}_{k_i,\ell_j} \in \Delta,\ \forall (i,j)\in E,. \tag{5h_{ij}}$$

It is not hard to see that (4) and (5) are equivalent by simply eliminating the variables $\{\lambda^j_{\ell_j}\}_{\ell_j,j}, \{\gamma^i_{k_i}\}_{k_i,i}$ in (5). However, this formulation will be much more convenient to work with for the proof of Theorem 1, as we outline next. Before doing so, we introduce some important shorthand together with some notation. Throughout the rest of the paper, for a given set of variables $X = \{X_1,\dots,X_n\}$, we associate $C_{(\text{eq. number})}(X)$, which corresponds to a set of constraints in (eq. number) involving at least one variable in X. Of interest to us will be the set

$$\{X \mid C_{(\text{eq. number})}(X)\}. \tag{6}$$

This set has two different meanings depending on the situation: either all constraints appearing in $C_{(\text{eq. number})}(X)$ involving *only* variables in X, in which case the set is well-defined; or there are some constraints appearing in $C_{(\text{eq. number})}(X)$ which also involve variables that are *not* in X, in which case the set means $\{X \mid \exists \bar{X} \text{ s.t. } C_{(\text{eq. number})}(X)\}$, where $\bar{X}$ are the variables appearing in (eq. number) which are not in X. To make this concrete, consider (5) and first let $X = \left\{\{\gamma^j_{k_j}\}_{k_j}, \{\lambda^j_{\ell_j}\}_{\ell_j}, \{r^j_{k_j\ell_j}\}_{k_j,\ell_j}\right\}$ for some $j \in \{1,\dots,n\}$. Let $C_{(5)}(X) = \{(5d_j),(5f_j),(5g_j)\}$. We then have:

$$\left\{X \mid C_{(5)}(X)\right\} = \Big\{\{\gamma^j_{k_j}\}_{k_j}, \{\lambda^j_{\ell_j}\}_{\ell_j}, \{r^j_{k_j\ell_j}\}_{k_j,\ell_j} \mid \lambda^j_{\ell_j} = \sum_{k_j=0}^{\Gamma_j} r^j_{k_j\ell_j},\ \forall \ell_j = 0,\dots,M,\ \gamma^j_{k_j} = \sum_{\ell_j=0}^{M} r^j_{k_j\ell_j},$$
$$\forall k_j = 0,\dots,\Gamma_j,\ \{r^j_{k_j\ell_j}\}_{k_j,\ell_j} \in \Delta\Big\}$$

Note that here $(5d_j)$, e.g., only appears for one specific $j \in \{1,\dots,n\}$ in $\left\{X \mid C_{(5)}(X)\right\}$. This is a setting where all constraints in $C_{(5)}(X)$ involve only variables in X. Now suppose that $X = \left\{\{\lambda^j_{\ell_j}\}_{\ell_j}\right\}$

for some $j \in \{1,\dots,n\}$ and let $C_{(5)}(X) = \{(5d_j), (5g_j)\}$. This is a setting where all constraints in $C_{(5)}(X)$ *do not* only involve variables in X. We then have:

$$\{X \mid C_{(5)}(X)\} = \left\{\{\lambda^j_{\ell_j}\}_{\ell_j} \mid \exists \{r^j_{k_j\ell_j}\}_{k_j,\ell_j} \text{ s.t. } \lambda^j_{\ell_j} = \sum_{k_j=0}^{\Gamma_j} r^j_{k_j\ell_j},\ \forall \ell_j = 0,\dots,M,\ \{r^j_{k_j\ell_j}\}_{k_j,\ell_j} \in \Delta\right\}$$

We now consider the feasible set P_{lin} of (5). Note that using the convention above, and letting

$$X = \left\{\{\lambda^j_{\ell_j}\}_{\ell_j,j}, \{\gamma^j_{k_j}\}_{k_j,j}, \{r^j_{k_j\ell_j}\}_{k_j,\ell_j,j}, \{s^{ij}_{k_i\ell_j}\}_{k_i,\ell_j,j}\right\}$$

and taking

$$C_{(5)}(X) = \{(5a_j), (5d_j), (5f_j), (5g_j), \forall j, (5b_{ij}), (5h_{ij}), \forall (i,j) \in E, (5c_j), \forall j \text{ s.t. } (i,j) \in E, (5e_i), \forall i \text{ s.t. } (i,j) \in E\},$$

we can write:

$$P_{lin} = \{X \mid C_{(5)}(X)\} \tag{7}$$

We break down the proof of Theorem 1 into three key steps.

Step 1: Understanding the structure of the tree decomposition of G'. Though it might not seem so at first glance, the structure of the tree decomposition $\mathcal{T}' = (T, \{X_z\}_{z\in T})$ of G' dictates our approach to solving Problem (2). We show that $\mathcal{T}'$ has treewidth 1 and that every bag contains either a pair (S_i, SI_j) for $i \neq j$ or a pair (S_j, SI_j) for some j (see Proposition EC.1). The structure of these bags lies behind the introduction of the variables $\{s^{ij}_{k_i\ell_j}\}$ corresponding to the former type of bags and the variables $\{r^j_{k_j\ell_j}\}$ corresponding to the latter type, as we discuss later.

Step 2: Binarization of Problem (2). The next step is to binarize Problem (2). This involves replacing SI_j and S_j by binary variables for $j = 1,\dots,m$. That is, we introduce binary variables $\{\lambda^j_{\ell_j}\}$ for $\ell_j = 0,\dots,M$ and $j = 1,\dots,n$ and $\{\gamma^j_{k_j}\}$ for $k_j = 0,\dots,\Gamma_j$ and $j = 1,\dots,n$, such that:

$$SI_j = \sum_{\ell_j=0}^{M} \ell_j \cdot \lambda^j_{\ell_j} \text{ and } S_j = \sum_{k_j=0}^{\Gamma_j} k_j \cdot \gamma^j_{k_j}, \text{ with } \{\lambda^j_{\ell_j}\}_{\ell_j} \in \Delta,\ \{\gamma^j_{k_j}\}_{k_j} \in \Delta \text{ for } j = 1,\dots,n.$$

In this setting, $\lambda^j_{\ell_j} = \mathbf{1}_{SI_j=\ell_j}$ and $\gamma^j_{k_j} = \mathbf{1}_{S_j=k_j}$, i.e., $\lambda^j_{\ell_j}$ and $\gamma^j_{k_j}$ play the role of indicator functions of the variables SI_j and S_j taking on values ℓ_j and k_j. Throughout the draft, we will work with the same convention: ℓ_j indexes the values taken on by SI_j and k_j indexes the values taken on by S_j. These are placed as subscripts of the variable. In the superscript appears the index j of the variable which is referred to. We write e.g. $\{\lambda^j_{\ell_j}\}_{\ell_j,j}$ to indicate that there is a new binary variable λ for each value ℓ_j taken on by each initial integer variable SI_j.

Each bag of the tree decomposition $\mathcal{T}'$ of G' contains exactly 2 variables and reflects the variables which we need to keep track of simultaneously. Thus, as mentioned above, we introduce more binary variables $\{s^{ij}_{k_i\ell_j}\}_{k_i\ell_j}$ for $(i,j)\in E$ (associated to the bags containing a pair (S_i, SI_j) in the tree) and $\{r^j_{k_j\ell_j}\}_{k_j\ell_j}$ for $j=1,\ldots,n$ (associated to the bags containing a pair (S_j, SI_j) in the tree). By properties of the tree decomposition, each pair (S_i, SI_j) (resp. (S_j, SI_j)) for $(i,j)\in E$ (resp. for $j=1,\ldots,n$) features at least once in one of the bags of the tree decomposition. These variables denote respectively the indicator variables of the pair (S_i, SI_j) taking on values (k_i,ℓ_j) *conjointly* and the indicator variables of the pair (S_j, SI_j) taking on values (k_j,ℓ_j) conjointly, once again. With these intepretations of the variables, it is easy to see that these constraints should hold:

$$\lambda^j_{\ell_j} = \sum_{k_j=0}^{\Gamma_j} r^j_{k_j\ell_j} = \sum_{k_i=0}^{\Gamma_i} s^{ij}_{k_i\ell_j},\ \forall \ell_j, j \text{ and } \gamma^i_{k_i} = \sum_{\ell_i=0}^{M} r^i_{k_i\ell_i} = \sum_{\ell_j=1}^{M} s^{ij}_{k_i\ell_j}, \forall k_i, i.$$

We then work on translating constraints ($2a_j$) and ($2b_{ij}$) into constraints that involve these binary variables. Likewise, we rewrite the objective function of (2) as:

$$\sum_{j=1}^{n}\sum_{k_j,\ell_j} f_j(\ell_j - k_j)\cdot r^j_{k_j\ell_j}.$$

With these points in mind, we can then show that (2) is equivalent to (5) (and thus (4)) *provided that all variables involved in* (5) *are integral*. This is formally shown in Proposition EC.3. The next difficulty is showing that the binary constraints can be dropped, that is that P_{lin} is integral.

Step 3: Showing that P_{lin} is integral. The overarching idea here is to break P_{lin}, the feasible set of (5), into smaller polytopes using the properties of $\mathcal{T}'$ in such a way that the join (see Definition 3) of these polytopes is P_{lin} itself. The goal will then be to show that the smaller polytopes are integral and that the join preserves integrality under certain assumptions.

Step 3a: Breaking P_{lin} into smaller polytopes which are integral. Based on the observations made in Step 1, we split the nodes in T into two sets: the set of nodes which contains (S_j, SI_j) and the set of nodes which contains (S_i, SI_j) for $i\neq j$. For a given $z\in T$, letting j_z be the index of the variable SI_j appearing in the set and i_z be that of S_i, we define:

$$T_1 = \{z\in T \mid i_z = j_z\} \text{ and } T_2 = \{z\in T \mid i_z \neq j_z\}. \tag{8}$$

Then, to each $z \in T$, we associate the binary counterpart of X_z, which we denote by Y_z. For $z \in T_1$, we let $X_z^1 = \{S_{j_z}, SI_{j_z}\}$, so Y_z^1 is defined as:

$$Y_z^1 = \left\{ \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \{\gamma_{k_{j_z}}^{j_z}\}_{k_{j_z}}, \{r_{k_{j_z}\ell_{j_z}}^{j_z}\}_{k_{j_z},\ell_{j_z}} \right\}, \tag{9}$$

i.e., Y_z^1 contains the indicator functions of S_{j_z}, SI_{j_z} and also the indicator function of the pair (S_{j_z}, SI_{j_z}). For $z \in T_2$, we have $X_z = \{S_{i_z}, SI_{j_z}\}$, so Y_z^2 is defined as:

$$Y_z^2 = \left\{ \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \{\gamma_{k_{i_z}}^{i_z}\}_{k_{i_z}}, \{s_{k_{i_z}\ell_{j_z}}^{ij}\}_{k_{i_z},\ell_{j_z}} \right\}, \tag{10}$$

i.e., Y_z^2 contains the indicator functions of S_{i_z}, SI_{j_z} and also the indicator function of the pair (S_{i_z}, SI_{j_z}). Likewise, for $z \in T_1$, we let $C_{(5)}(Y_z^1) = \{(5a_{j_z}), (5d_{j_z}), (5f_{j_z}), (5g_{j_z})\}$ and when $z \in T_2$, we let $C_{(5)}(Y_z^2) = \{(5b_{i_z j_z}), (5c_{j_z}), (5e_{i_z}), (5h_{i_z j_z})\}$. These two elements enable us to define the polytopes for $z \in T$:

$$Q_z^1 = \{Y_z^1 \mid C_{(5)}(Y_z^1)\} \text{ for } z \in T_1, \text{ and } Q_z^2 = \{Y_z^2 \mid C_{(5)}(Y_z^2)\} \text{ for } z \in T_2. \tag{11}$$

By properties of the tree decomposition described in Step 1, one can see that

$$P_{lin} = \left(\wedge_{z \in T_1} Q_z^1 \right) \wedge \left(\wedge_{z \in T_2} Q_z^2 \right).$$

Furthermore, it can be shown that Q_z^i is integral for $i \in \{1,2\}$ and any $z \in T$ (Lemma EC.7).

Step 3b: Showing that the join operation preserves integrality in this case. We show that the set $\wedge_{z \in T} Q_z$ is integral using a specific sequence of enumeration of the nodes in T and properties of the join. The fact that P_{lin} is integral follows, as does the theorem.

We refer the reader to Appendix B for the rigorous proof.

REMARK 1. As discussed above, Section 4 considers a more general set-up where G is not a tree, but rather a graph with bounded treewidth ω. This leads to some major changes in the proof. In Step 1, the structure of $\mathcal{T}'$ changes significantly. In particular, the width of the decomposition becomes ω: this implies that we need to keep track of sets of variables of size ω rather than 2 as done here. This leads to the introduction of new binary decision variables and constraints, though they remain similar in flavor to those appearing in (5). These changes, in turn, leads to changes in Step 3, with different polytopes to consider, though, again, the approach remains largely unchanged.

4. Solving the GSM on a network G with treewidth ω

In this section, we show how to solve Problem 2 for general graphs of treewidth ω. This extends the approach of Humair and Willems (2006), who focus on "supply chains with clusters of commonality." Starting with tree graphs, individual nodes are replaced by bipartite subgraphs. Their algorithm runs in time $O(n_T M^c)$, where n_T is the number of nodes in the starting tree, and c is the number of nodes on the "small side" of the largest bipartite subgraph. Note that, in a complete bipartite subgraph with a nodes on one side and b nodes on the other side, the treewidth is exactly $\omega = \min\{a, b\}$. Hence, we can say that the algorithm by Humair and Willems (2006) runs in time $O(n_T M^\omega)$. For ω to be bounded, we further need that $n_T \sim n$, that is, we can upper-bound the runtime by $O(nM^\omega)$.

4.1. Properties of a tree decomposition of the intersection graph of G with treewidth ω

As in Section 3, we show that Problem 2 can be rewritten as a linear program, the difference here being that its size scales with the treewidth ω of G. To be able to write down this linear program, we need some structural information on the tree decomposition of G', which we give now.[3]

PROPOSITION 2. *Let G be a directed graph of treewidth ω and let G' be its intersection graph obtained as described in Proposition 1. There exists a tree decomposition $\mathcal{T} = (T, \{X_z\}_{z\in T})$ of G' with the following properties: (i) $\mathcal{T}$ has width at most $\omega + 1$; (ii) every bag X_z contains exactly one variable SI_j for some $j \in \{1, \ldots, n\}$.*

The proof of this proposition is deferred to Appendix C.1. We now consider a setting where such a tree decomposition $\mathcal{T} = (T, \{X_z\}_{z\in T})$ is available to us and we introduce some notation relating to this decomposition before stating the main theorem. To illustrate, we use the example given in Figure 2a. The intersection graph corresponding to P is given in Figure 2b, with a tree decomposition as described in Proposition 2 given in Figure 2c. The tree underlying the tree decomposition is given in Figure 2d. Consider the tree decomposition $\mathcal{T} = (T, \{X_z\}_{z\in T})$ of G' as given in Proposition 2. We define for any $z \in T$,

$$I_z = \{i \in \{1, \ldots, n\} \mid S_i \in X_z\} \text{ and } J_z = \{j \in \{1, \ldots, n\} \mid SI_j \in X_z\}.$$

Note that from Proposition 2, $|J_z| = 1$ and $|I_z| + |J_z| \leq \omega + 2$. We let $J_z = \{j_z\}$. In Figure 2c, $J_4 = \{3\}$ and $I_4 = \{2, 3\}$. We also introduce a partition of the nodes in T as in Section 3, into T_1 and T_2:

$$T_1 = \{z \in T \mid j_z \in I_z\} \text{ and } T_2 = \{z \in T \mid j_z \notin I_z\}. \tag{12}$$

[3] In terms of the proof structure given in Section 3.3, a difference between Section 3 and this section is that we need to cover Step 1 to state Theorem 2.

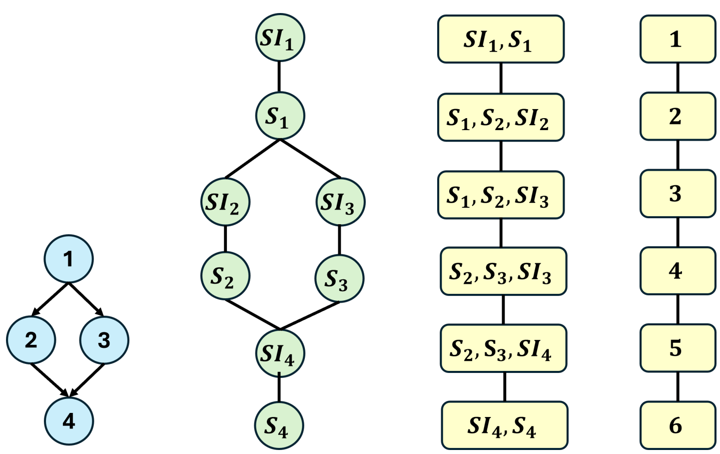
In our example, $T_1 = \{1,2,4,6\}$ and $T_2 = \{3,5\}$.

We introduce a new set of decision variables compared to the tree case in Section 3. More specifically, we maintain the variables appearing in (5), that is, $\{\lambda^j_{\ell_j}\}_{\ell_j,j}, \{\gamma^j_{k_j}\}_{k_j,j}$, which are involved in the binarization of the variables $\{SI_j\}_j$ and $\{S_j\}_j$ for $j = 1,\ldots,n$, and $\{r^j_{k_j\ell_j}\}_{k_j,\ell_j,j}$, which corresponds to the indicator function of whether $(S_j, SI_j) = (k_j, \ell_j)$ for $j = 1,\ldots,n$. In addition to these, we have binary decision variables which keep track of the values taken on by all the variables in each *bag* of T:[4] for $z \in T$, we denote these variables by $\left\{s^{\{S_i\}_{i\in I_z} SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}\right\}^z_{\{k_i\}_{i\in I_z},\ell_{j_z}}$. In other words, in our running example, in node 4, with $I_4 = \{2,3\}$ and $J_4 = \{3\}$, we consider the variable $s^{S_2S_3SI_3}_{k_2k_3\ell_3}$. The superscript keeps track of the variables that are contained within the bag X_z. The subscript corresponds to the value taken on by the variable: S_i takes values $k_i = 0,\ldots,\Gamma_i$ for $i \in I_z$. For fixed $z \in T$, we have $\prod_{i\in I_z}(\Gamma_i + 1)\cdot(M+1)$ variables $\left\{s^{\{S_i\}_{i\in I_z} SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}\right\}^z_{\{k_i\}_{i\in I_z},\ell_{j_z}}$. As $\Gamma_i \le M$ for all $i = 1,\ldots,n$,, there are at most $\omega + 2$ variables per bag (see Proposition 2), and a tree decomposition of an intersection graph with $2n$ variables has $8n$ bags at most, the number of variables considered in the next theorem is at most $2n(M+1) + n(M+1)^2 + 8n(M+1)^{\omega+2} = O(nM^\omega)$.

We can now define the counterpart of $\mathcal{S}_{ij}$ in (3) for the general treewidth case:

$$\mathcal{S}_{i_zj_z} = \left\{((k_i)_{i\in I_z}, \ell_{j_z}) \in \times_{i\in I_z}\{0,\ldots,\Gamma_i\} \times \{0,\ldots,M\} \mid k_{i_z} > \ell_{j_z}\right\},\ \forall (i_z, j_z) \in (I_z \times J_z) \cap E, z \in T. \tag{13}$$

In Figure 2c, when $z = 5$, two sets are defined, namely S^5_{24} and S^5_{34} as $(2,4)$ and $(3,4)$ are in E.



(a) G (b) G' (c) $\mathcal{T}$ (d) T

Figure 2 **An example where G is a 4-cycle, together with its intersection graph G', and its tree decomposition $\mathcal{T} = (T, \{X_z\}_{z\in T})$ with T also represented.**

[4] In Section 3, the variables $\{r^j_{k_j\ell_j}\}_{k_j,\ell_j,j}$ and $\{s^{ij}_{k_j\ell_j}\}_{k_j,\ell_j,(i,j)}$ conjointly played this role.

4.2. Main theorem of the section

With the notation introduced previously, we are able to state the main theorem in this section.

THEOREM 2. *When $G = (V, E)$ is a graph of treewidth ω, the guaranteed service model given in (2) can be formulated as a linear program in $O(nM^{\omega})$ variables and $O(nM^{\omega})$ constraints. If $\mathcal{T} = (T, \{X_z\}_{z\in T})$ is a tree decomposition of the intersection graph G' as described in Proposition 2 and with the notation given in Section 4.1, this linear program can be written as:*

$$\min \quad \sum_{j=1}^{n} \sum_{\ell_j=0}^{M} \sum_{k_j=0}^{\Gamma_j} f_j(\ell_j - k_j) r^j_{k_j \ell_j} \tag{14}$$

$$\text{s.t.} \quad r^j_{k_j\ell_j} \le \left\lfloor \frac{T_j}{k_j - \ell_j} \right\rfloor, \ \forall k_j = 0,\dots,\Gamma_j,\ \ell_j = 0,\dots,M,\ k_j > \ell_j, j = 1,\dots,n \tag{$14a_j$}$$

$$s^{\{S_i\}_{i\in I_z} S_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}} = 0,\ \forall((k_i)_{i\in I_z}, \ell_{j_z}) \in \mathcal{S}_{i_z j_z},\ (i_z, j_z) \in (I_z, J_z) \cap E,\ z \in T \tag{$14b^z_{i_z j_z}$}$$

$$r^{j_z}_{k_{j_z}\ell_{j_z}} = \sum_{\{k_i \mid i\in I_z, i\neq j_z\}} s^{\{S_i\}_{i\in I_z} S_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}},\ \forall k_{j_z} = 0,\dots,\Gamma_{j_z}, \forall \ell_{j_z} = 0,\dots,M,\ \forall z \in T_1, \tag{$14c^z$}$$

$$\lambda^{j_z}_{\ell_{j_z}} = \sum_{\{k_i \mid i\in I_z\}} s^{\{S_i\}_{i\in I_z} S_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}},\ \forall \ell_{j_z} = 0,\dots,M,\ \forall z \in T, \tag{$14d^z$}$$

$$\gamma^{i_z}_{k_{i_z}} = \sum_{\{k_i \mid i\in I_z, i\neq i_z\}} \sum_{\ell_{j_z}} s^{\{S_i\}_{i\in I_z} S_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}}, \forall k_{i_z} = 0,\dots,\Gamma_{i_z},\ \forall i_z \in I_z,\ \forall z \in T, \tag{$14e^z_{i_z}$}$$

$$\left\{ s^{\{S_i\}_{i\in I_z} SI_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}} \right\}^z_{\{k_i\}_{i\in I_z}, \ell_{j_z}} \in \Delta,\ \forall z \in T, \tag{$14f^z$}$$

with decision variables $\{\lambda^j_{\ell_j}\}_{\ell_j,j}, \{\gamma^j_{k_j}\}_{k_j,j}, \{r^j_{k_j\ell_j}\}_{k_j,\ell_j,j}, \left\{ s^{\{S_i\}_{i\in I_z} SI_{j_z}}_{\{k_i\}_{i\in I_z} \ell_{j_z}} \right\}^z_{\{k_i\}_{i\in I_z}, \ell_{j_z}}$.

The proof of the theorem can be found in Appendix C. It follows a similar outline to the proof of Theorem 1 given in Section 3.3, though Step 1 has already been covered in Section 4.1.

REMARK 2. The linear program in (14) is the counterpart of (5): constraint ($5a_j$) corresponds to ($14a_j$), ($5b_{ij}$) to ($14b^z_{i_z j_z}$), ($5c_j$)-($5d_j$) is subsumed by ($14d^z$), ($5e_i$)-($5f_j$) by ($14e^z_{i_z}$), and finally ($5g_j$)-($5h_{ij}$) by ($14f^z$). One can make (14) more compact, in a similar way that one can reduce (5) to (4), by getting rid of the variables $\{\lambda^j_{\ell_j}\}, \{\gamma^j_{k_j}\}$. However, this complicates the notation significantly, which is why we do not present such a linear program here. We do give both formulations for the running example in Figure 2a next, however.

For our running example in Figure 2a, we introduce four new binary decision variables: $s^{S_1S_2SI_2}_{k_1k_2\ell_2}, s^{S_1S_2SI_3}_{k_1k_2\ell_3}, s^{S_2S_3SI_3}_{k_2k_3\ell_3}, s^{S_2S_3SI_4}_{k_2k_3\ell_4}$ corresponding to nodes 2,3,4,5 in the tree T. Nodes 1 and 6 in T

only contain two variables, so do not require the introduction of new binary decision variables. The linear program is then given by:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{4}\sum_{\ell_j=0}^{M}\sum_{k_j=0}^{\Gamma_j} f_j(\ell_j-k_j) r^j_{\ell_j k_j} \\
\text{s.t.} \quad & r^j_{k_j\ell_j} \le \left\lfloor \frac{T_j}{k_j-\ell_j} \right\rfloor, \ \forall k_j=0,\dots,\Gamma_j,\ \ell_j=0,\dots,M,\ k_j>\ell_j, j=1,\dots,4 \\
& s^{S_1S_2SI_2}_{k_1k_2\ell_2}=0, \forall k_1>\ell_2, \forall k_2,\ s^{S_2S_3SI_3}_{k_2k_3\ell_3}=0, \forall k_2>\ell_3, \forall k_2, \\
& s^{S_1S_2SI_3}_{k_1k_2\ell_3}=0, \forall k_1>\ell_3, \forall k_2,\ s^{S_1S_2SI_3}_{k_1k_2\ell_3}=0, \forall k_2>\ell_3, \forall k_1, \\
& s^{S_2S_3SI_4}_{k_2k_3\ell_4}=0, \forall k_2>\ell_4, \forall k_3,\ s^{S_2S_3SI_4}_{k_2k_3\ell_4}=0, \forall k_3>\ell_4, \forall k_2, \\
& r^2_{k_2\ell_2}=\sum_{k_1} s^{S_1S_2SI_2}_{k_1k_2\ell_2},\ \forall k_2,\ell_2,\ \ r^3_{k_3\ell_3}=\sum_{k_2} s^{S_2S_3SI_3}_{k_2k_3\ell_3},\ \forall k_3,\ell_3, \\
& \gamma^1_{k_1}=\sum_{k_2}\sum_{\ell_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2}, \forall k_1,\ \ \gamma^2_{k_2}=\sum_{k_1}\sum_{\ell_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2}, \forall k_2, \\
& \gamma^1_{k_1}=\sum_{k_2}\sum_{\ell_3} s^{S_1S_2SI_3}_{k_1k_2\ell_3}, \forall k_1,\ \ \gamma^2_{k_2}=\sum_{k_1}\sum_{\ell_3} s^{S_1S_2SI_3}_{k_1k_2\ell_3}, \forall k_2, \\
& \gamma^2_{k_2}=\sum_{k_3}\sum_{\ell} s^{S_2S_3SI_3}_{k_2k_3\ell_3}, \forall k_2,\ \ \gamma^3_{k_3}=\sum_{k_2}\sum_{\ell_3} s^{S_2S_3SI_3}_{k_2k_3\ell_3}, \forall k_3, \\
& \gamma^2_{k_2}=\sum_{k_3}\sum_{\ell} s^{S_2S_3SI_4}_{k_2k_3\ell_4}, \forall k_2,\ \ \gamma^3_{k_3}=\sum_{k_2}\sum_{\ell_4} s^{S_2S_3SI_4}_{k_2k_3\ell_4}, \forall k_3, \\
& \lambda^2_{\ell_2}=\sum_{k_1}\sum_{k_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2}, \forall \ell_2,\ \lambda^3_{\ell_3}=\sum_{k_1}\sum_{k_2} s^{S_1S_2SI_3}_{k_1k_2\ell_3},\ \forall \ell_3, \\
& \lambda^3_{\ell_3}=\sum_{k_2}\sum_{k_3} s^{S_2S_3SI_3}_{k_2k_3\ell_3}, \forall \ell_3,\ \lambda^4_{\ell_4}=\sum_{k_2}\sum_{k_3} s^{S_2S_3SI_4}_{k_2k_3\ell_4},\ \forall \ell_4, \\
& s^{S_1S_2SI_2}_{k_1k_2\ell_2}\in\Delta, s^{S_1S_2SI_3}_{k_1k_2\ell_3}\in\Delta, s^{S_2S_3SI_3}_{k_2k_3\ell_3}\in\Delta, s^{S_2S_3SI_4}_{k_2k_3\ell_4}\in\Delta, \\
& r^1_{k_1\ell_1}\in\Delta, r^2_{k_4\ell_4}\in\Delta.
\end{aligned}
$$

It is easy to substitute the decision variables $\{\lambda^j_{\ell_j}\}, \{\gamma^j_{k_j}\}$ in the previous optimization problem to obtain a more compact formulation:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{4}\sum_{\ell_j=0}^{M}\sum_{k_j=0}^{\Gamma_j} f_j(\ell_j-k_j) r^j_{\ell_j k_j} \\
\text{s.t.} \quad & r^j_{k_j\ell_j} \le \left\lfloor \frac{T_j}{k_j-\ell_j} \right\rfloor, \ \forall k_j=0,\dots,\Gamma_j,\ \ell_j=0,\dots,M,\ k_j>\ell_j, j=1,\dots,4 \\
& s^{S_1S_2SI_2}_{k_1k_2\ell_2}=0, \forall k_1>\ell_2, \forall k_2,\ s^{S_2S_3SI_3}_{k_2k_3\ell_3}=0, \forall k_2>\ell_3, \forall k_2,
\end{aligned}
$$

$$s^{S_1S_2SI_3}_{k_1k_2\ell_3} = 0, \forall k_1 > \ell_3, \forall k_2,\ s^{S_1S_2SI_3}_{k_1k_2\ell_3} = 0, \forall k_2 > \ell_3, \forall k_1,$$
$$s^{S_2S_3SI_4}_{k_2k_3\ell_4} = 0, \forall k_2 > \ell_4, \forall k_3,\ s^{S_2S_3SI_4}_{k_2k_3\ell_4} = 0, \forall k_3 > \ell_4, \forall k_2,$$
$$r^2_{k_2\ell_2} = \sum_{k_1} s^{S_1S_2SI_2}_{k_1k_2\ell_2},\ \forall k_2, \ell_2,\ \ r^3_{k_3\ell_3} = \sum_{k_2} s^{S_2S_3SI_3}_{k_2k_3\ell_3},\ \forall k_3, \ell_3,$$
$$\sum_{k_2}\sum_{\ell_3} s^{S_1S_2SI_3}_{k_1k_2\ell_3} = \sum_{k_2}\sum_{\ell_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2}, \forall k_1,$$
$$\sum_{k_1}\sum_{\ell_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2} = \sum_{k_1}\sum_{\ell_3} s^{S_1S_2SI_3}_{k_1k_2\ell_3} = \sum_{k_3}\sum_{\ell} s^{S_2S_3SI_3}_{k_2k_3\ell_3} = \sum_{k_3}\sum_{\ell} s^{S_2S_3SI_4}_{k_2k_3\ell_4}, \forall k_2,$$
$$\sum_{k_2}\sum_{\ell_3} s^{S_2S_3SI_3}_{k_2k_3\ell_3} = \sum_{k_2}\sum_{\ell_4} s^{S_2S_3SI_4}_{k_2k_3\ell_4}, \forall k_3,$$
$$\sum_{k_2}\sum_{k_3} s^{S_2S_3SI_3}_{k_2k_3\ell_3} = \sum_{k_1}\sum_{k_2} s^{S_1S_2SI_3}_{k_1k_2\ell_3},\ \forall \ell_3,$$
$$s^{S_1S_2SI_2}_{k_1k_2\ell_2} \in \Delta, s^{S_1S_2SI_3}_{k_1k_2\ell_3} \in \Delta, s^{S_2S_3SI_3}_{k_2k_3\ell_3} \in \Delta, s^{S_2S_3SI_4}_{k_2k_3\ell_4} \in \Delta,$$
$$r^1_{k_1\ell_1} \in \Delta, r^2_{k_4\ell_4} \in \Delta.$$

5. The GSM in closed-loop supply chain networks

We now allow for the reverse flow of materials, that is, we will no longer assume that G is acyclic. First, we show how we can adjust the GSM to appropriately compute the inventory costs in such cases. While Minner (2001) provides a smiliar model accounting for reverse flows, we approach the extension differently, based on the framework of Graves and Willems (2000). This is because the former also requires special consideration of undirected cycles, which the latter directly includes, thus simplifying the derivation.

Second, we argue that the procedure we developed to solve the GSM on networks with treewidth ω directly extends to this extension, without affecting the complexity (as long as ω is unchanged). Importantly, we highlight that extending a graph G by adding individual reverse flows increases ω at most by 1, supporting the usefulness of our procedure to systematically analyze circular networks.

5.1. A simple example.

We will use the example in Figure 3 to clarify the ideas behind the extension.

We assume that the production of unit of output by node 5 also leads to a secondary product. This secondary product can serve as an input to node 2, thus replacing the necessary inputs from node 1. In particular, we will assume that the secondary product from the production of one unit of node 5's output can replace $\eta \leq 1$ units of input from node 1.[5] Note that the inbound service time for

[5] We could, alternatively, assume that the secondary product from node 5 is an additional input that is necessary, besides the input from node 1. This would only marginally affect the analysis, but we would need to modify our assumption that each output requires one unit of each input, an assumption made for simpler exposition.

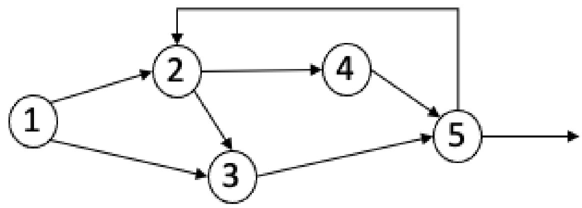


Figure 3 **Example of a case with a directed cycle**

node 2, SI_2 cannot directly capture the input received from node 5 as in the baseline formulation of the GSM, because we need to consider different cases. Finally, we assume that all of the secondary product is actually used.

Next, we consider in detail how many units of inventory are incoming and outgoing at each node. First, consider node 5:

$$I_5(t) = B_5 - \sum_{m=1}^{t-S_5} D_5^m + \sum_{m=1}^{t-T_5-SI_5} D_5^m.$$

This is unchanged from the baseline case, and the base stock and relevant costs are as before.

Similarly, the computation is unchanged for nodes 4 and 3. Next, consider node 2. Here, the inventory equation changes:

$$\begin{aligned} I_2(t) =& B_2 - \sum_{m=1}^{t-S_2} 2D_5^m + \sum_{m=1}^{t-T_2-SI_2} (2-\eta)D_5^m + \sum_{m=1}^{\min\{t-S_2, t-T_2-S_5\}} \eta D_5^m \\ &+ \sum_{m=\min\{t-S_2, t-T_2-S_5\}}^{t-T_2-S_5} \eta D_5^m. \end{aligned}$$

The first sum again represents the demand to be covered. Due to the structure of the network, whenever node 5 observes D_5^t units of demand, this causes it to order that quantity in inputs from both node 3 and node 4. Hence, either of the latter nodes will start producing D_5^t of their products, in turn triggering demand for inputs from node 2. As a result, node 2 needs to produce $2D_5^m$ units to cover the external demand.

The second sum represents the output produced using input from node 1. All of the secondary input is used, so demand for x units of output from node 2 will cause it to trigger an order of $(x-\eta)$ inputs from node 1. Note that, although demand exceeding the upper bound is dealt with outside the model, we assume that this demand still triggers the relevant secondary product.

The third and fourth sums represent the output produced using node 5's secondary product as an alternative input. Importantly, input from node 5 can only be processed after production at node

5, that is, after S_5 units of time (to which the processing time T_2 needs to be added until they are available as outputs).

The fourth sum is relevant only if $T_2 + S_5 < S_2$. In this case, a demand at the end customer leads to production by node 5 so much earlier than by node 2, that the corresponding quantity of the secondary product from node 5 arrives in node 2's inventory before the latter can use it.

In total, we need to consider three cases:

1. $S_5 + T_2 < S_2 \leq SI_2 + T_2$:

$$I_2(t) = B_2 - \sum_{m=t-SI_2-T_2}^{t-S_2} (2-\eta) D_5^m + \sum_{m=t-S_2}^{t-T_2-S_5} \eta D_5^m.$$

   The final term covers the secondary product that arrives before it can be used to cover the relevant demand. Node 2 can use this inventory and adjust its base stock accordingly: $B_2' = B_2 - \sum_{m=t-S_2}^{t-T_2-S_5} \eta D_5^m$. To guarantee demand fulfillment within the outbound service time, node 2 needs to hold inventory to cover $(2-\eta)$ times the baseline (i.e., customer) demand across $SI_2 + T_2 - S_2$ periods.

2. $S_2 \leq S_5 + T_2 \leq SI_2 + T_2$:

$$I_2(t) = B_2 - \sum_{m=t-SI_2-T_2}^{t-S_2} 2D_5^m + \sum_{m=t-SI_2-T_2}^{t-T_2-S_5} \eta D_5^m.$$

   In this case, both the secondary and primary inputs are unavailable at the time of the demand. The secondary input arrives first. Hence, the base stock level needs to cover 2 times the baseline demand across $S_5 + T_2 - S_2$ periods, plus $(2-\eta)$ times the baseline demand across $SI_2 - S_5$ periods.

3. $S_2 \leq SI_2 + T_2 < S_5 + T_2$:

$$I_2(t) = B_2 - \sum_{m=t-SI_2-T_2}^{t-S_2} 2D_5^m - \sum_{m=t-T_2-S_5}^{t-SI_2-T_2} \eta D_5^m.$$

   In this case, again both types of inputs are unavailable at the time of demand. This time, the primary input arrives second. Hence, the base stock level needs to cover 2 times the baseline demand across $SI_2 + T_2 - S_2$ periods, plus η times the baseline demand across $S_5 - SI_2$ periods.

For example, if we assume that node 2 guarantees to fulfill all demand up to $\mu_2 + k\sigma_2$, and we further assume that $D_5^t \sim Normal(\mu_5, \sigma_5)$, then the cost at 2 can be computed as follows:

$$\begin{cases} h_j k\sigma_5\sqrt{(2-\eta)^2(SI_2+T_2-S_2)}, \text{ if } S_5+T_2 < S_2 \leq SI_2+T_2 \\ h_j k\sigma_5\sqrt{2^2(S_5+T_2-S_2)+(2-\eta)^2(SI_2-S_5)}, \text{ if } S_2 \leq S_5+T_2 \leq SI_2+T_2 \\ h_j k\sigma_5\sqrt{2^2(SI_2+T_2-S_2)+\eta^2(S_5-SI_2)}, \text{ if } S_2 \leq SI_2+T_2 < S_5+T_2. \end{cases}$$

Finally, for node 1, we need to adjust total demand, based on the replacement of node 1's outputs to node 2 by the secondary product. That is, the demand observed by node 1 at time t is $D_1^t = (1+2-\eta)D_5^t$. Using this demand, the computation of base stock follows as in the base case.

Remark 3. As the example shows, the reverse flow only affects the structure of the inventory cost function of the recipient node. The inventory costs of other nodes j may require careful reconfiguration to take into account the demand covered by reverse flows, but remain solely dependent on the decision variables SI_j and S_j. Hence, we can use the same derivation to allow for multiple reverse flows, as long as any single node only receives at most one such flow.

5.2. Solving the GSM for closed-loop supply chain networks.

Based on the previous derivation, we can formulate the GSM for closed-loop supply chain networks with a general objective function. Assume that, starting with an acylic graph $G = (V, E)$, individual reverse flows are added, forming an additional edge set E^*. That is, we consider a new graph $G_R = (V, E \cup E^*)$. We let $R = \{j \ : \ (i, j) \in E^*\}$ be the set of recipient nodes and i_j^* the outgoing node of the reverse flow ending in $j \in R$. We will assume that that $|R| = |E^*|$, that is, each node receives at most one reverse flow. We can then write

$$\begin{aligned} \min_{S_j, SI_j} \quad & \sum_{j \in V \setminus R} f_j(SI_j - S_j) + \sum_{j \in R} f_j(SI_j - S_j, S_{i_j^*} - S_j, SI_j - S_{i_j^*}) \\ \text{s.t.} \quad & S_j - SI_j \leq T_j, \ j = 1, \ldots, n \\ & SI_j - S_i \geq 0, \ (i, j) \in E \\ & S_j \leq s_j, \ j \in L \\ & S_j, SI_j \geq 0 \text{ and integer for } j = 1, \ldots, n. \end{aligned} \tag{14}$$

Our previously derived procedure extends directly. The only difference is that we need to account for the value of $S_{i_j^*}$ when deciding the values of S_j and SI_j. Hence, we need to extend the intersection graph accordingly, by adding the edges $(S_{i_j^*}, S_j)$ and $(S_{i_j^*}, SI_j)$ for each $j \in R$.

Say that the starting (acylic) graph G has treewidth ω, so that there is a tree decomposition $\mathcal{T}$ of its intersection graph G' with treewidth at most $\omega+1$ (Proposition 2). Let G_R be the extended graph with a single reverse flow (i^*_j, j) added for some $j \in V$, and G'_R its intersection graph. We will construct $\mathcal{T}_R$, a tree decomposition of G'_R, from $\mathcal{T}$. There already exists a bag in $\mathcal{T}$ containing SI_j and S_j. Either one such bag already contains $S_{i^*_j}$, or we need to add $S_{i^*_j}$ to all bags on the shortest path between any bag containing $S_{i^*_j}$ and any bag containing SI_j and S_j. Hence, $\mathcal{T}_R$ has treewidth at most $\omega+2$. Adding multiple reverse flows does not affect the treewidth of G'_R further, unless the shortest paths defined above intersect. Thus, in the worst case, the treewidth increases by one per reverse flow added.

References

Agrawal VV, Atasu A, Van Wassenhove LN (2019) OM Forum—New opportunities for operations management research in sustainability. *Manufacturing & Service Operations Management* 21(1):1–12.

Applegate D, Díaz M, Hinder O, Lu H, Lubin M, O'Donoghue B, Schudy W (2021) Practical large-scale linear programming using primal-dual hybrid gradient. *Advances in Neural Information Processing Systems* 34:20243–20257.

Billington C, Callioni G, Crane B, Ruark JD, Rapp JU, White T, Willems SP (2004) Accelerating the profitability of Hewlett-Packard's supply chains. *Interfaces* 34(1):59–72.

Blaettchen P, Calmon AP, Hall G (2024) Traceability technology adoption in supply chain networks. *Management Science* (Articles in Advance).

Bodlaender HL (1994) A tourist guide through treewidth. *Acta Cybern.* 11(1-2):1.

Calmon AP, Graves SC (2017) Inventory management in a consumer electronics closed-loop supply chain. *Manufacturing & Service Operations Management* 19(4):568–585.

Eruguz AS, Sahin E, Jemai Z, Dallery Y (2016) A comprehensive survey of guaranteed-service models for multi-echelon inventory optimization. *International Journal of Production Economics* 172:110–125.

Farasyn I, Humair S, Kahn JI, Neale JJ, Rosen O, Ruark J, Tarlton W, Van de Velde W, Wegryn G, Willems SP (2011) Inventory optimization at Procter & Gamble: Achieving real benefits through user adoption of inventory tools. *Interfaces* 41(1):66–78.

Graves SC, Willems SP (2000) Optimizing strategic safety stock placement in supply chains. *Manufacturing & Service Operations Management* 2(1):68–83.

Humair S, Willems SP (2006) Optimizing strategic safety stock placement in supply chains with clusters of commonality. *Operations Research* 54(4):725–742.

Klosterhalfen ST, Minner S, Willems SP (2014) Strategic safety stock placement in supply networks with static dual supply. *Manufacturing & Service Operations Management* 16(2):204–219.

Lesnaia E (2004) *Optimizing safety stock placement in general network supply chains*. Ph.D. thesis, Massachusetts Institute of Technology.

Manary MP, Wieland B, Willems SP, Kempf KG (2019) Analytics makes inventory planning a lights-out activity at Intel Corporation. *INFORMS Journal on Applied Analytics* 49(1):52–63.

Minner S (2001) Strategic safety stocks in reverse logistics supply chains. *International Journal of Production Economics* 71(1-3):417–428.

Schoenmeyr T, Graves SC (2022) Coordination of multiechelon supply chains using the guaranteed service framework. *Manufacturing & Service Operations Management* 24(3):1859–1871.

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Appendix A: Some useful Lemmas

The following definitions and results are used across the proofs of Theorems 1 and 2.

DEFINITION EC.1. Let $u_0, u_1, \ldots, u_n$ be $(n+1)$ affinely independent points. The corresponding simplex $\Delta_n(u_0, u_1, \ldots, u_n)$ is the n-dimensional set given by

$$\Delta_n(u_0, u_1, \ldots, u_n) = \left\{ \theta_0 u_0 + \theta_1 u_1 + \ldots + \theta_n u_n \mid \sum_{i=0}^{n} \theta_i = 1,\ \theta_i \geq 0, \forall i = 0, \ldots, n \right\}.$$

Note that if $u_0, \ldots, u_n$ are integral, then $\Delta(u_0, \ldots, u_n)$ is an integral polytope as it is simply the convex hull of $u_0, \ldots, u_n$. The *standard* simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$, as introduced in Section 3, refers to the setting where $u_0, \ldots, u_n$ are the $(n+1)$ standard unit vectors in $\mathbb{R}^{n+1}$.

LEMMA EC.1. *Let $\Delta_{n-1} \subseteq \mathbb{R}^n$ be the standard simplex and let $V \in \{0,1\}^{m \times n}$ be a binary matrix. Define*

$$Q := \{(x, y) \in \mathbb{R}^{n+m} \mid x \in \Delta_{n-1}, y = Vx\}.$$

Then, $Q \subseteq \mathbb{R}^{n+m}$ is an $(n-1)$-dimensional simplex with binary vertices $\begin{bmatrix} e_i \\ v_i \end{bmatrix}$ for $i = 1, \ldots, n$, where e_i is the i^{th} standard unit vector in $\mathbb{R}^n$ and v_i is the i^{th} column of V.

This result implies that Q thus defined is integral.

Proof. Let $u_i = \begin{bmatrix} e_i \\ Ve_i \end{bmatrix} = \begin{bmatrix} e_i \\ v_i \end{bmatrix}$ for $i = 1, \ldots, n$. Note that $u_i \in \{0,1\}^{n+m}$ as $V \in \{0,1\}^{m \times n}$. Furthermore, $u_1, \ldots, u_n$ are affinely independent due to the fact that $e_1, \ldots, e_n$ are affinely independent. We now show that $Q = \Delta(u_1, \ldots, u_n)$. If $(x, y) \in Q$, then there exist $\theta_1, \ldots, \theta_n$ such that $\sum_{i=1}^n \theta_i = 1,\ \theta_i \geq 0,\ \forall i = 1, \ldots, n$, and $x = \sum_{i=1}^n \theta_i e_i$ as $x \in \Delta_{n-1}$. We then have $y = Vx = \sum_{i=1}^n \theta_i V e_i$, which implies that $(x, y) = \sum_{i=1}^n \theta_i u_i$ and $(x, y) \in \Delta(u_1, \ldots, u_n)$. Conversely, if $(x, y) \in \Delta(u_1, \ldots, u_n)$, then there exist $\theta_1, \ldots, \theta_n$ such that $\sum_{i=1}^n \theta_i = 1, \theta_i \geq 0,\ \forall i = 1, \ldots, n$, and $(x, y) = \sum_{i=1}^n \theta_i u_i$. It follows from this that $x = \sum_{i=1}^n \theta_i e_i$, i.e., $x \in \Delta_{n-1}$ and $y = \sum_{i=1}^n \theta_i V e_i = Vx$. This in turn implies that $(x, y) \in Q$. □

LEMMA EC.2. *Let $u_0, u_1, \ldots, u_k \in \{0,1\}^n$ be affinely independent vectors for $k \leq n$. Let $P = \Delta_k(u_0, \ldots, u_k)$ and let $S \subseteq \{1, \ldots, n\}$. Then,*

$$Q := \{x \in \mathbb{R}^n \mid x \in P, x_i = 0, \forall i \in S\}$$

is a simplex whose vertices are a subset of $u_0, u_1, \ldots, u_k \in \{0,1\}^n$.

Proof. Let $S_1 = \{k' \in \{0, \ldots, k\} \mid u_k^i = 0, \forall i \in S\}$. In other words, S_1 contains the indexes of a subset of the vertices of P for which all i^{th} entries are zero for any $i \in S$. As $\{u_{k'}\}_{k' \in S_1}$ is a subset of $u_0, \ldots, u_k$, this set remains affinely independent. Letting $Q' = \Delta_{|S_1|-1}(\{u_{k'}\}_{k' \in S_1})$, we show that $Q' = Q$. This proves that Q is a simplex. It is straightforward to see that $Q' \subseteq Q$. For $Q \subseteq Q'$, let $x \in Q$: we have that $x = \sum_{k'=0}^k \theta_{k'} u_{k'}$ where $\sum_{k'=0}^k \theta_{k'} = 1$ and $\theta_{k'} \geq 0, \forall k' = 0, \ldots, k$. Let $k_0 \notin S_1$. By definition, there exists $i_{k_0} \in S$ such that $u_{k_0}^{i_{k_0}} \neq 0$, i.e., $u_{k_0}^{i_{k_0}} = 1$ as $u_0, \ldots, u_k$ are binary. Thus,

$$x_{i_{k_0}} = \sum_{k' \in S_1} \theta_{k'} u_{k'}^{i_{k_0}} + \sum_{k' \notin S_1} \theta_{k'} u_{k'}^{i_{k_0}} = \sum_{k' \notin S_1} \theta_{k'} u_{k'}^{i_{k_0}} = \theta_{k_0} + \sum_{k' \notin S_1, k' \neq k_0} \theta_{k'} u_{k'}^{i_{k_0}} = 0.$$

As $\theta_{k_0} \geq 0$ and $+\sum_{k' \notin S_1, k' \neq k_0} \theta_{k'} u_{k'}^{i_{k_0}} \geq 0$, it must be the case that $\theta_{k_0} = 0$. We repeat this procedure for any $k' \notin S_1$. It follows that $\theta_{k'} = 0, \forall k' \notin S_1$ and thus $x \in Q'$. □

LEMMA EC.3. *Let $u_0, u_1, \ldots, u_k \in \{0,1\}^n$ be affinely independent vectors for $k \leq n$. Let $P = \Delta_k(u_0, \ldots, u_k)$ and let $S' \subseteq \{1, \ldots, n\}$. Further, let $b_i \in \{0, 1, \ldots\}$ for any $i \in S'$. Then, $Q' := \{x \in \mathbb{R}^n \mid x \in P, x_i \leq b_i, \forall i \in S'\}$ is a simplex whose vertices are a subset of $u_0, u_1, \ldots, u_k \in \{0,1\}^n$.*

Proof. This is a straightforward corollary of Lemma EC.2 by taking $S = \{i \in S' \mid b_i = 0\}$. Then, note that $Q' = \{x \in \mathbb{R}^n \mid x \in P, x_i = 0, \forall i \in S\}$. This is due to the fact that the vertices of P are binary, so we have that $0 \leq x_i \leq 1$ for any $i = 1, \ldots, n$ from $x \in P$. Thus, if $b_i \geq 1$, $x_i \leq b_i$ is a redundant constraint and if $b_i = 0$, then $x_i = 0$. □

Recall the definition of the projection of a polytope $Q \subseteq \mathbb{R}^{n+m}$ onto the variables $x \in \mathbb{R}^n$:

$$proj_x(Q) = \left\{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m \text{ s.t. } \begin{pmatrix} x \\ y \end{pmatrix} \in Q \right\}.$$

LEMMA EC.4. *Let $P \subseteq R^n$ be an integral polytope and let $Q \subseteq R^{m+n}$ be an integral polytope. If for any integral $x \in P$, $\exists$ an integral y such that $\begin{pmatrix} x \\ y \end{pmatrix} \in Q$, and conversely, if for any integral $\begin{pmatrix} x \\ y \end{pmatrix} \in Q$, $x \in P$, then $P = proj_x(Q)$.*

Proof. Let $x \in P$. As P is an integral polytope, there exist integral vertices $v_1, \ldots, v_N$ of P such that $x = \sum_{i=1}^N \lambda_i v_i$, where $\sum_{i=1}^N \lambda_i = 1$ and $\lambda_i \geq 0$, $\forall i = 1, \ldots, N$. As v_i is integral and in P, there exists y_i integral such that $\begin{pmatrix} v_i \\ y_i \end{pmatrix} \in Q$. As Q is convex, $\sum_{i=1}^N \lambda_i \begin{pmatrix} v_i \\ y_i \end{pmatrix} = \begin{pmatrix} x \\ \sum_{i=1}^N \lambda_i y_i \end{pmatrix} \in Q$. Thus $x \in proj_x(Q)$. Now, let $x \in proj_x(Q)$, i.e., there exists y such that $\begin{pmatrix} x \\ y \end{pmatrix} \in Q$. As Q is an integral polytope, we can write $\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=1}^M \gamma_i \begin{pmatrix} u_i \\ w_i \end{pmatrix}$ where $\begin{pmatrix} u_i \\ w_i \end{pmatrix}$ are integral and in Q, and $\sum_{i=1}^M \gamma_i = 1, \gamma_i \geq 0, \forall i = 1, \ldots, M$. Thus $u_i \in P$ and, as P is convex, $x = \sum_{i=1}^M \gamma_i u_i \in P$. □

LEMMA EC.5 **(Lemma APDX.5 in Kim et al. 2022)**. *For $i = 1, 2$, let Q_i be an integral polytope in variables $(x_i, y) \in \mathbb{R}^{p_i} \times \mathbb{R}^{p_0}$. Assume that $proj_y(Q_1) = proj_y(Q_2) = \Delta$, where Δ is some simplex. Then the join $Q_1 \wedge Q_2$ as defined in Definition 3 is integral.*

We wrap up this section with a result relating to tree decompositions (see Definition 2) of an intersection graph. For a node z in T, we let $par(z)$ be its unique parent in T.

LEMMA EC.6. *Let G' be an intersection graph as defined in Definition 1 and let $\mathcal{T}' = (T, \{X_z\}_{z \in T})$ be a tree decomposition of G' of width ω. There exists a labeling $(1, \ldots, T_0)$ of the nodes of T such that $(X_1 \cup \ldots \cup X_{i-1}) \cap X_i = X_{par(i)} \cap X_i$ for all $i = 2, \ldots, T_0 - 1$.. In particular, the overlap is of size at most ω.*

Proof. Pick a node of T to be the root and then label the nodes using, e.g., the level-order traversal. This ensures that the labels of the parents are before those of the children. As a child in a tree has a unique parent, node i is connected via exactly one edge to the subtree generated by $(1, \ldots, i)$: namely i is connected to its parent. The properties of a tree decomposition give us that $(X_1 \cup \ldots \cup X_{i-1}) \cap X_i = X_{par(i)} \cap X_i$. The fact that one can always find a tree decomposition with distinct bags (otherwise the two identical bags can be merged) implies that the overlap is of size at most ω as each bag contains $\omega + 1$ variables. □

Appendix B: Proof of Theorem 1

PROPOSITION EC.1. *Let G be a directed tree and let G' be its intersection graph obtained as described in Proposition 1. There exists a tree decomposition $\mathcal{T}' = (T, \{X_z\}_{z \in T})$ of G' with the following properties: (i) $\mathcal{T}'$ has width 1; (ii) every bag contains either a pair (S_i, SI_j) for $i \neq j$ or a pair (S_j, SI_j) for some $j \in \{1, \ldots, n\}$.*

Proof. As G is a directed tree, it is not difficult to see that G' is also a tree, with edges between SI and S variables only. Build a tree decomposition $\mathcal{T}' = (T, \{X_z\}_{z\in T})$ of G' in the following way: for any edge in G', introduce a node z in T and place in the corresponding bag X_z the variables in G' that this edge connects. Add an edge to T between two nodes z and z' if X_z and $X_{z'}$ share a variable. One can easily see that such a construction is a valid tree decomposition. Furthermore, it has width 1 as there are two variables in each bag. As there are only edges in G' between SI and S variables, it follows that the bags in T contain exactly one S variable and one SI variable. □

Before proceeding with Steps 2 and 3 as described in Section 3.3, we introduce some notation. Recall the definitions of T_1 and T_2 in (8), of the sets of binary variables Y_z^1 and Y_z^2 in (9) and (10), and the polytopes Q_z^1 and Q_z^2 in (11). We now define subsets of variables of Y_z^1 and Y_z^2, which correspond to sets of variables that can appear in the overlap of two bags Y_z and $Y_{par(z)}$, where $par(z)$ is the unique parent of node z in the tree. We further associate polytopes to these subsets of variables, which are themselves subsets of Q_z^1 and Q_z^2. We also introduce the counterparts of these notions in the space of variables $\{S_i\}$ and $\{SI_j\}$, in a sense that we make precise in Proposition EC.2. To make these connections precise, we present this notation in Table EC.1.

	Integer Space (SI, S)	Binary Space (λ, γ, r, s)
Bags	$X_z^1 = \{S_{j_z}, SI_{j_z}\}$	$Y_z^1 = \left\{\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}, \{\gamma^{j_z}_{k_{j_z}}\}_{k_{j_z}}, \{r^{j_z}_{k_{j_z}\ell_{j_z}}\}_{k_{j_z},\ell_{j_z}}\right\}$
in T_1	$R_z^1 = \{X_z^1 \mid (2a_{j_z}), (2c_{j_z}), (2d_{j_z})\}$	$Q_z^1 = \{Y_z^1 \mid (5a_{j_z}), (5d_{j_z}), (5f_{j_z}), (5g_{j_z})\}$
Bags	$X_z^2 = \{S_{i_z}, SI_{j_z}\}$	$Y_z^2 = \left\{\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}, \{\gamma^{i_z}_{k_{i_z}}\}_{k_{j_z}}, \{s^{i_z j_z}_{k_{i_z}\ell_{j_z}}\}_{k_{i_z},\ell_{j_z}}\right\}$
in T_2	$R_z^2 = \{X_z^1 \mid (2b_{i_z j_z}), (2c_{i_z}), (2d_{j_z})\}$	$Q_z^2 = \{Y_z^1 \mid \{(5b_{i_z j_z}), (5c_{j_z}), (5e_{i_z}), (5h_{i_z j_z})\}$
Overlaps - Case 1	$\tilde{R}_z^1 = \{SI_{j_z} \mid (2d_{j_z})\}$	$\tilde{Y}_z^1 = \{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}$ and $\tilde{Q}_z^1 = \{\tilde{Y}_z^1 \mid (5d_{j_z}), (5g_{j_z})\}$
Overlaps - Case 2	$\tilde{R}_z^2 = \{S_{j_z} \mid (2c_{j_z})\}$	$\tilde{Y}_z^2 = \{\gamma^{j_z}_{k_{j_z}}\}_{k_{j_z}}$ and $\tilde{Q}_z^2 = \{\tilde{Y}_z^2 \mid (5f_{j_z}), (5g_{j_z})\}$
Overlaps - Case 3	$\tilde{R}_z^3 = \{SI_{j_z} \mid (2d_{j_z})\}$	$\tilde{Y}_z^3 = \{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}$ and $\tilde{Q}_z^3 = \{\tilde{Y}_z^3 \mid (5c_{j_z}), (5h_{i_z j_z})\}$
Overlaps - Case 4	$\tilde{R}_z^4 = \{S_{i_z} \mid (2c_{i_z})\}$	$\tilde{Y}_z^4 = \{\gamma^{i_z}_{k_{i_z}}\}_{k_{i_z}}$ and $\tilde{Q}_z^4 = \{\tilde{Y}_z^4 \mid (5e_{i_z}), (5h_{i_z j_z})\}$

Table EC.1 Sets of variables and polytopes of interest in the proofs of Theorem 1.

We now show results relating to these polytopes. In the reminder, we let $(1, \ldots, T_0)$ be an ordering of the nodes of T as described in Lemma EC.6.

Proposition EC.2. *We have:*

(i) For any $i \in \{1, \ldots, 4\}$ and any $z \in T$, there exists a point in $\tilde{R}_z^i$ if and only if there exists an integral point in $\tilde{Q}_z^i$.

(ii) For any $z \in T$, there exists a point in $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} R_z^1\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} R_z^2\right)$ if and only if there exists an integral point in $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} Q_z^1\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} Q_z^2\right)$.

Proof. We show (i) first. If $\tilde{R}_z^1$ is non-empty, then take $\lambda^{j_z}_{\ell_{j_z}} = \mathbf{1}_{SI_{j_z}=\ell_{j_z}}$ for any ℓ_{j_z}, where $\mathbf{1}$ is the indicator function. Clearly, $\lambda_{\ell_{j_z}}$ is integral and is in $\tilde{Q}_z^1$ (take, e.g., $r^{j_z}_{0\ell_{j_z}} = 1$ for all ℓ_{j_z} and $r^{j_z}_{k_{j_z}\ell_{j_z}} = 0$ for $k_{j_z} \neq 0$). Conversely, if $\lambda_{\ell_{j_z}}$ is in $\tilde{Q}_z^1$, then $\lambda_{\ell_{j_z}} \in \Delta$, where Δ is the standard simplex. As $\{\lambda_{\ell_{j_z}}\}_{\ell_{j_z}}$ is integral, this implies that there exists $\tilde{\ell}_{j_z} \in \{0, \ldots, M\}$

such that $\lambda_{\tilde{\ell}_{jz}} = 1$ and $\lambda_{\ell_{jz}} = 0$ for any $\ell_{jz} \neq \tilde{\ell}_{jz}$. Taking $SI_{jz} = \ell_{jz}$, we get that $\tilde{R}^1_z$ is non-empty. For $(i,j) = (1,2)$, we proceed similarly: If $\tilde{R}^2_z$ is non-empty, then take $\gamma^{jz}_{k_{jz}} = \mathbf{1}_{S_{jz}=k_{jz}}$ for any k_{jz}. Clearly, $\gamma_{k_{jz}}$ is integral and is in $\tilde{Q}^2_z$ (take, e.g., $s^{i_z j_z}_{k_{jz}0} = 1$ for all k_{jz} and $s^{i_z j_z}_{k_{iz}\ell_{jz}} = 0$ for $k_{iz} \neq 0$). Conversely, if $\gamma_{k_{jz}}$ is in $\tilde{Q}^2_z$, then $\gamma_{k_{jz}} \in \Delta$. As $\{\gamma_{k_{jz}}\}_{k_{jz}}$ is integral, this implies that there exists $\tilde{k}_{jz} \in \{0,\dots,\Gamma_{jz}\}$ such that $\gamma_{\tilde{k}_{jz}} = 1$ and $\gamma_{k_{jz}} = 0$ for any $k_{jz} \neq \tilde{k}_{jz}$. Taking $S_{jz} = k_{jz}$, we get that $\tilde{R}^2_z$ is non-empty. We proceed in exactly the same way for $i \in \{3,4\}$.

We now show (ii). If $\left(\wedge_{z'\in\{1,\dots,z\}\cap T_1} R^1_z\right) \wedge \left(\wedge_{z'\in\{1,\dots,z\}\cap T_2} R^2_z\right)$ is non-empty, then take

$$\begin{aligned}
\lambda^{j_{z'}}_{\ell_{j_{z'}}} &= \mathbf{1}_{SI_{j_{z'}}=\ell_{j_{z'}}}, \forall \ell_{j_{z'}} = 0,\dots,T,\ z' \in \{1,\dots,z\},\\
\gamma^{i_{z'}}_{k_{i_{z'}}} &= \mathbf{1}_{S_{i_{z'}}=k_{i_{z'}}}, \forall k_{i_{z'}} = 0,\dots,\Gamma_{i_{z'}},\ z' \in \{1,\dots,z\},\\
r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}} &= \mathbf{1}_{S_{j_{z'}}=k_{j_{z'}},SI_{j_{z'}}=\ell_{j_{z'}}},\ \forall k_{j_{z'}},\ell_{j_{z'}},\ z' \in \{1,\dots,z\}\cap T_1,\\
s^{i_{z'}j_{z'}}_{k_{i_{z'}},\ell_{j_{z'}}} &= \mathbf{1}_{S_{i_{z'}}=k_{i_{z'}},SI_{j_{z'}}=\ell_{j_{z'}}},\ \forall k_{i_{z'}},\ell_{j_{z'}},\ z' \in \{1,\dots,z\}\cap T_2.
\end{aligned}$$

These are binary variables, thus integer. Let $z' \in \{1,\dots,z\}\cap T_1$. By construction, ($5d_{j_{z'}}$) and ($5f_{j_{z'}}$) hold as summing the indicator function of two variables over one of the variables returns the indicator function of the other variable, as does ($5g_{j_{z'}}$). For constraint ($5a_{j_{z'}}$), let $k_{j_{z'}} > \ell_{j_{z'}}$. There are two options. Either $r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}} = 0$, in which case ($5a_{j_{z'}}$) trivially holds as $T_{j_{z'}} \geq 0$ and $k_{j_{z'}} - \ell_{j_{z'}} \geq 0$. Or, $r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}} = 1$, in which case, as $r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}} = \mathbf{1}_{S_{j_{z'}}=k_{j_{z'}},SI_{j_{z'}}=\ell_{j_{z'}}}$, ($2a_{j_{z'}}$) implies that $k_{j_{z'}} - \ell_{j_{z'}} \leq T_{j_{z'}}$, which in turn implies that $1 \leq \frac{T_{j_{z'}}}{k_{j_{z'}}-\ell_{j_{z'}}}$ and that ($5a_{j_{z'}}$) holds. Now, let $z' \in \{1,\dots,z\}\cap T_2$. By construction, ($5c_{j_{z'}}$) and ($5e_{i_{z'}}$) hold as does ($5h_{i_{z'}j_{z'}}$). For ($5b_{i_{z'}j_{z'}}$), suppose that ($2b_{i_{z'}j_{z'}}$) holds for $(S_{i_{z'}}, SI_{j_{z'}})$. Moreover, suppose that $SI_{j_{z'}} = \ell_{j_{z'}}$ and $S_{i_{z'}} = k_{i_{z'}}$ for some $k_{i_{z'}},\ell_{j_{z'}}$. As $SI_{j_{z'}} - S_{i_{z'}} \geq 0$, we must have that $\ell_{j_{z'}} \geq k_{i_{z'}}$. Thus, it cannot be that $k_{i_{z'}} > \ell_{j_{z'}}$ and so $s^{i_{z'}j_{z'}}_{k_{i_{z'}},\ell_{j_{z'}}} = 0$ when $k_{i_{z'}} > \ell_{j_{z'}}$. Thus, ($5b_{i_{z'}j_{z'}}$) holds given the definition of $\mathcal{S}_{ij}$ in (3). This implies that there exists an integral point in $\left(\wedge_{z'\in\{1,\dots,z\}\cap T_1} Q^1_z\right) \wedge \left(\wedge_{z'\in\{1,\dots,z\}\cap T_2} Q^2_z\right)$.

Now suppose that $\left(\wedge_{z'\in\{1,\dots,z\}\cap T_1} Q^1_z\right) \wedge \left(\wedge_{z'\in\{1,\dots,z\}\cap T_2} Q^2_z\right)$ contains an integral point. For $z' \in \{1,\dots,z\}\cap T_1$, as $Y^1_{z'}$ is integral and ($5d_{j_{z'}}$), ($5f_{j_{z'}}$), ($5g_{j_{z'}}$) hold, we have that $\{\lambda^{j'_z}_{\ell_{j_{z'}}}\}_{\ell_{j_{z'}}}$ is binary, as is $\{\gamma^{i_{z'}}_{k_{j_{z'}}}\}_{k_{j_{z'}}}$. For $z' \in \{1,\dots,z\}\cap T_2$, as $Y^2_{z'}$ is integral and ($5c_{j_{z'}}$), ($5e_{i_{z'}}$), ($5h_{i_{z'}j_{z'}}$) hold, we have that $\{\lambda^{j'_z}_{\ell_{j_{z'}}}\}_{\ell_{j_{z'}}}$ is binary, as is $\{\gamma^{i_{z'}}_{k_{i_{z'}}}\}_{k_{i_{z'}}}$. Thus, for any $z' \in \{1,\dots,z\}$, we can define:

$$S_{i_{z'}} = \sum_{k_{i_{z'}}=0}^{\Gamma_{i_{z'}}} k_{i_{z'}} \cdot \gamma^{i_{z'}}_{k_{i_{z'}}} \quad \text{and} \quad SI_{j_{z'}} = \sum_{\ell_{j_{z'}}=0}^{M} \ell_{j_{z'}} \cdot \lambda^{j_{z'}}_{\ell_{j_{z'}}}.$$

That ($2c_{j_{z'}}$) and ($2d_{j_{z'}}$) hold is immediate for any $z' \in \{1,\dots,z\}$. Now, let $z' \in \{1,\dots,z\}\cap T_1$. For constraint ($2a_{j_{z'}}$), constraint ($5g_{j_{z'}}$) together with the fact that $r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}}$ is integer (and thus binary) implies that there exist $\hat{\ell}_{j_{z'}} \in \{0,\dots,M\}$ and $\hat{k}_{j_{z'}} \in \{0,\dots,\Gamma_{j_{z'}}\}$ such that $r^{j_{z'}}_{\hat{k}_{j_{z'}},\hat{\ell}_{j_{z'}}} = 1$ with $r^{j}_{k_{j_{z'}},\ell_{j_{z'}}} = 0$ for any $(k_{j_{z'}},\ell_{j_{z'}}) \neq (\hat{k}_{j_{z'}},\hat{\ell}_{j_{z'}})$. Coupled with ($5d_{j_{z'}}$) and ($5f_{j_{z'}}$), this implies that $S_{j_{z'}} = \hat{k}_{j_{z'}}$ and $SI_{j_{z'}} = \hat{\ell}_{j_{z'}}$. If $\hat{k}_{j_{z'}} \leq \hat{\ell}_{j_{z'}}$, then $S_{j_{z'}} - SI_{j_{z'}} = \hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}} \leq T_{j_{z'}}$ trivially holds as $T_{j_{z'}} \geq 0$. Now suppose that $\hat{k}_{j_{z'}} > \hat{\ell}_{j_{z'}}$. Constraint ($5a_{j_{z'}}$) implies that $r_{\hat{k}_{j_{z'}},\hat{\ell}_{j_{z'}}} \leq T_{j_{z'}}/(\hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}})$, i.e., $(\hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}}) = S_{j_{z'}} - SI_{j_{z'}} \leq T_{j_{z'}}$, which is ($2a_{j_{z'}}$). Now, let $z' \in \{1,\dots,z\}\cap T_2$. From ($5c_{j_{z'}}$) and ($5f_{j_{z'}}$), we have that:

$$SI_{j_{z'}} - S_{i_{z'}} = \sum_{\ell_{j_{z'}}=0}^{M} \ell_{j_{z'}} \cdot \lambda^{j_{z'}}_{\ell_{j_{z'}}} - \sum_{k_{i_{z'}}=0}^{\Gamma_{i_{z'}}} k_{i_{z'}} \cdot \gamma^{i_{z'}}_{k_{i_{z'}}} = \sum_{\ell_{j_{z'}}=0}^{M} \sum_{k_{i_{z'}}=0}^{\Gamma_{i_{z'}}} (\ell_{j_{z'}} - k_{i_{z'}}) s^{i_{z'}j_{z'}}_{k_{i_{z'}},\ell_{j_{z'}}}.$$

From the definition of $\mathcal{S}_{i_{z'}j_{z'}}$, constraint ($2b_{i_{z'}j_{z'}}$) implies that if $k_{i_{z'}} > \ell_{j_{z'}}$, $s^{i_{z'}j_{z'}}_{k_{i_{z'}},\ell_{j_{z'}}} = 0$. Thus, $SI_{j_{z'}} - S_{i_{z'}} \geq 0$, which is ($2b_{i_{z'}j_{z'}}$). □

PROPOSITION EC.3. *Problems (5) and (2) are equivalent provided that the variables appearing in (5) are constrained to be integers.*

Proof. Recall that by definition of the tree decomposition of G', for any $j \in \{1, \ldots, n\}$, there exists a node $z \in T_1$ such that $(S_{j_z}, SI_{j_z}) \in X_z$ and for any $(i, j) \in E$, there exists a node $z \in T_2$ such that $(S_{i_z}, SI_{j_z}) \in X_z$. The proof then follows immediately from Proposition EC.2 (ii) with $z = T_0$, further noting that the objective functions of (2) and (5) are equivalent with the choice of variables made in the proof of the proposition. □

LEMMA EC.7. *The polytope Q_z^1 is integral for any $z \in T_1$ and the polytope Q_z^2 is integral for any $z \in T_2$.*

Proof. Let $z \in T_2$. From Table EC.1, we have:

$$Q_z^2 = \Big\{\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}, \{\gamma^{i_z}_{k_{i_z}}\}_{k_{i_z}}, \{s^{i_z j_z}_{k_{i_z}\ell_{j_z}}\}_{k_{i_z},\ell_{j_z}} \mid s^{i_z j_z}_{k_{i_z}\ell_{j_z}} = 0,\ \forall (k_{i_z}, \ell_{j_z}) \in \mathcal{S}_{i_z j_z},\ \lambda^{j_z}_{\ell_{j_z}} = \sum_{k_{i_z}=0}^{\Gamma_{i_z}} s^{ij}_{k_{i_z}\ell_{j_z}},\ \forall \ell_{j_z},$$
$$\gamma^{i_z}_{k_{i_z}} = \sum_{\ell_{j_z}=0}^{M} s^{i_z j_z}_{k_{i_z}\ell_{j_z}},\ \forall k_{i_z},\ \{s^{i_z j_z}_{k_{i_z}\ell_{j_z}}\}_{k_{i_z}\ell_{j_z}} \in \Delta\Big\}.$$

Dropping the constraints $s^{i_z j_z}_{k_{i_z}\ell_{j_z}} = 0,\ \forall (k_{i_z}, \ell_{j_z}) \in \mathcal{S}_{i_z j_z}$ from Q_z^2, it is not difficult to see that the new set can be written:

$$\Big\{\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}, \{\gamma^{i_z}_{k_{i_z}}\}_{k_{i_z}}, \{s^{i_z j_z}_{k_{i_z}\ell_{j_z}}\}_{k_{i_z},\ell_{j_z}} \mid \lambda^{j_z} = V_{j_z} s^{i_z j_z},\ \gamma^{i_z} = W_{i_z} s^{i_z j_z},\ s^{i_z j_z} \in \Delta\Big\},$$

where λ^{j_z} (resp. $\gamma^{i_z}, s^{i_z j_z}$) are vectors consisting of the variables $\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}$ (resp. $\{\gamma^{i_z}_{k_{i_z}}\}_{k_{i_z}}, \{s^{i_z j_z}_{k_{i_z}\ell_{j_z}}\}_{k_{i_z},\ell_{j_z}}$) stacked, and V_{j_z} and W_{i_z} are some binary matrices. Thus, following Lemma EC.1, it is a simplex with binary vertices. Adding back on the constraints $s^{i_z j_z}_{k_{i_z}\ell_{j_z}} = 0,\ \forall (k_{i_z}, \ell_{j_z}) \in \mathcal{S}_{i_z j_z}$, we obtain a simplex with binary vertices once again (see Lemma EC.2). Thus Q_z^2 is integral when $z \in T_2$.

Suppose now that $z \in T_1$. We can write:

$$Q_z^1 = \Big\{\{\lambda^{j_z}_{\ell_{j_z}}\}_{\ell_{j_z}}, \{\gamma^{j_z}_{k_{j_z}}\}_{k_{j_z}}, \{r^{j_z}_{k_{j_z}\ell_{j_z}}\}_{k_{j_z},\ell_{j_z}} \mid r^{j_z}_{k_{j_z}\ell_{j_z}} \le \left\lfloor \frac{T_{j_z}}{k_{j_z} - \ell_{j_z}} \right\rfloor \forall k_{j_z} > \ell_{j_z},\ \lambda^{j_z}_{\ell_{j_z}} = \sum_{k_{j_z}=0}^{\Gamma_{j_z}} r^{j_z}_{k_{j_z}\ell_{j_z}},\ \forall \ell_{j_z},$$
$$\gamma^{j_z}_{k_{j_z}} = \sum_{\ell_{j_z}=0}^{M} r^{j_z}_{k_{j_z}\ell_{j_z}},\ \forall k_{j_z},\ \{r^{j_z}_{k_{j_z}\ell_{j_z}}\}_{k_{j_z}\ell_{j_z}} \in \Delta\Big\}.$$

Similarly to above, if we drop the constraints $r^{j_z}_{k_{j_z}\ell_{j_z}} \le \left\lfloor \frac{T_{j_z}}{k_{j_z} - \ell_{j_z}} \right\rfloor \forall k_{j_z} > \ell_{j_z}$ from Q_z^1, we obtain a simplex with binary vertices following Lemma EC.1. Adding back the constraints, we obtain once again a simplex with binary vertices, following Lemma EC.3. Thus Q_z^1 is integral for any $z \in T_1$. □

PROPOSITION EC.4. *Let $\tilde{Q}_z^i$ for $i \in \{1, \ldots, 4\}$ and Q_z^i for $i \in \{1, 2\}$ be the sets defined in Table EC.1. We have:*

(i) If $\tilde{Q}_z^1$ or $\tilde{Q}_z^2$ is nonempty, then Q_z^1 is nonempty.

(ii) If $\tilde{Q}_z^3$ or $\tilde{Q}_z^4$ is nonempty, then Q_z^2 is nonempty.

(iii) If $\tilde{Q}_z^i$ is nonempty for some $i \in \{1, \ldots, 4\}$ then $\left(\wedge_{z' \in \{1,\ldots,par(z)\} \cap T_1} Q^1_{z'}\right) \wedge \left(\wedge_{z' \in \{1,\ldots,par(z)\} \cap T_2} Q^2_{z'}\right)$ is nonempty.

Proof. We show instead the following statements:

(i) If $\tilde{R}_z^1$ or $\tilde{R}_z^2$ is nonempty, then R_z^1 is nonempty.

(ii) If $\tilde{R}_z^3$ or $\tilde{R}_z^4$ is nonempty, then R_z^2 is nonempty.

(iii) If $\tilde{R}_z^i$ is nonempty for some $i \in \{1, \ldots, 4\}$ then $\left(\wedge_{z' \in \{1,\ldots,par(z)\}\cap T_1} R_{z'}^1\right) \wedge \left(\wedge_{z' \in \{1,\ldots,par(z)\}\cap T_2} R_{z'}^2\right)$ is nonempty.

The proposition follows immediately from Proposition EC.2.

We now show $(i)-(ii)$. To show (i), assume that $\tilde{R}_z^1$ is nonempty. Then take $S_{j_z}=0$ and R_z^1 is nonempty. If $\tilde{R}_z^2$ is nonempty, take $SI_{j_z}=S_{j_z}$ and R_z^1 is nonempty. To show (ii), if $\tilde{R}_z^3$ is nonempty, take $S_{i_z}=0$ and R_z^2 is nonempty. If $\tilde{R}_z^4$ is nonempty, take $SI_{j_z}=S_{i_z}$ and R_z^2 is nonempty.

Assuming $(i)-(ii)$ hold, we now show (iii). Let

$$W_{z_0} = \left(\{\wedge_{z' \in \{par_{z_0}(z),\ldots,par(z)\}\cap T_1} R_{z'}^1\right) \wedge \left(\{\wedge_{z' \in \{par_{z_0}(z),\ldots,par(z)\}\cap T_2} R_{z'}^2\right)$$

where $par_{z_0}(z)$ corresponds to the node obtained after applying the parent operation to z, z_0 times. We show the following by induction on z_0:

$$\text{if } \tilde{R}_z^i \text{ is nonempty for some } i \in \{1,\ldots,4\}, \text{ then } W_{z_0} \text{ is nonempty for } z_0 \geq 1. \tag{EC.1}$$

From Lemma EC.6, this is equivalent to (iii) when $z_0 = z-1$. Suppose first that $z_0=1$. If $par(z) \in T_1$, then $W_{z_0} = R^1_{par(z)}$; if $par(z) \in T_2$, then $W_{z_0} = R^2_{par(z)}$. Assume that $\tilde{R}_z^i$ is non-empty for some $i \in \{1,\ldots,4\}$. From $(i)-(ii)$, then either R_z^1 (if $z \in T_1$) or R_z^2 (if $z \in T_2$) is non-empty. It cannot be that z and $par(z)$ are both in T_1: if they are, their overlap is the empty set, and the initial graph G must have isolated nodes. Let us assume now that R_z^1 is non-empty and $par(z) \in T_2$. The overlap is either $\{SI_{j_z}\}$ or $\{S_{j_z}\}$. In the former case, as $SI_{j_z} = SI_{j_{par(z)}}$, it follows that $\tilde{R}^3_{par(z)}$ is non-empty, and thus from (ii), $R^2_{par(z)} = W_{z_0}$ is non-empty. In the latter case, $S_{j_z} = S_{j_{par(z)}}$, it follows that $\tilde{R}^4_{par(z)}$ is nonempty and thus from (ii), $R^2_{par(z)} = W_{z_0}$ is nonempty. We show a similar result when R_z^2 is nonempty and $par(z) \in T$. Now suppose that (EC.1) holds for z_0-1. We show it holds for z_0. We have:

$$W_{z_0} = W_{par(z_0-1)} = W_{z_0-1} \wedge R^1_{par_{z_0}(z)} \text{ if } par_{z_0}(z) \in T_1$$

and

$$W_{z_0} = W_{par(z_0-1)} = W_{z_0-1} \wedge R^2_{par_{z_0}(z)} \text{ if } par_{z_0}(z) \in T_2.$$

From (EC.1), W_{z_0-1} is non-empty. Let $\cup_{z' \in \{z,\ldots,par_{z_0-1}(z)\}} X_{z'}$ be the point in W_{z_0-1}. From Lemma EC.6, $X_{par_{z_0}(z)} \cap \left(\cup_{z' \in \{z,\ldots,par_{z_0-1}(z)\}} X_{z'}\right) = X_{par_{z_0}(z)} \cap X_{par_{z_0-1}(z)}$ and the overlap is at most one. If $par_{z_0}(z) \in T_1$, then this overlap is either $SI_{j_{par_{z_0}(z)}}$ or $S_{j_{par_{z_0}(z)}}$. In the first case, necessarily, $\tilde{R}^1_{par_{z_0}(z)}$ is non-empty as $SI_{j_{par_{z_0}(z)}} \in W_{z_0-1}$. In the second case, likewise, $\tilde{R}^2_{par_{z_0}(z)}$ is non-empty as $S_{j_{par_{z_0}(z)}} \in W_{z_0-1}$. Thus (i) applies and $R^1_{par(z)}$ is nonempty as is W_{z_0}. We proceed similarly if $par_{z_0}(z) \in T_2$ to conclude that W_{z_0} is non-empty. This proves the result. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We show by induction on $z \in (1,\ldots,T_0)$ that

$$U_z := \left(\wedge_{z' \in \{1,\ldots,z\}\cap T_1} Q_z^1\right) \wedge \left(\wedge_{z' \in \{1,\ldots,z\}\cap T_2} Q_z^2\right)$$

is integral. By properties of the tree decomposition of the intersection graph G', it is clear that $U_{T_0} = P_{lin}$. Hence, our induction shows that P_{lin} is integral and the theorem follows from Proposition EC.3.

Suppose that $z=1$. If $1 \in T_1$, then Lemma EC.7 shows that Q_1^1 is integral; if $1 \in T_2$ then Lemma EC.7 shows that Q_1^2 is integral. Thus, U_1 is integral. Now suppose that $U_{par(z)}$ is integral and consider U_z. We have $U_z = U_{par(z)} \wedge Q_z^1$ if $z \in T_1$ and $U_z = U_{par(z)} \wedge Q_z^2$ if $z \in T_2$. Our goal is to use Lemma EC.5 to show that U_z is integral. The induction hypothesis gives us that $U_{par(z)}$ is integral and Q_z^i is integral from Lemma EC.7 for any $i \in \{1,2\}$. Thus it remains to show that the projection of $U_{par(z)}$ and Q_z^i for $i = \{1,2\}$ onto their common variables is a common simplex. This is what we show now.

The variables appearing in Q_z^i are Y_z^i for $i \in \{1,2\}$ and the variables appearing in $U_{par(z)}$ are the variables $\left(\cup_{z' \in \{1,\ldots,par(z)\}\cap T_1} Y_{z'}^1\right) \cup \left(\cup_{z' \in \{1,\ldots,par(z)\}\cap T_2} Y_{z'}^2\right)$. From Lemma EC.6, the common variables between the two sets are then $Y_{par(z)}^i \cap Y_z^j$ for $i,j \in \{1,2\}$. A simple check reveals that this intersection can only be $\tilde{Y}_z^1, \tilde{Y}_z^2$, $\tilde{Y}_z^3$, and $\tilde{Y}_z^4$, depending on whether $z \in T_1$ or $z \in T_2$.

If $z \in T_1$, then either the overlap is $\tilde{Y}_z^1$ or it is $\tilde{Y}_z^2$. If it is $\tilde{Y}_z^1$, then it is easy to see that

$$proj_{\tilde{Y}_z^1} Q_z^1 = \Delta = proj_{\tilde{Y}_z^1} U_{par(z)}.$$

Indeed, for the first equality, $proj_{\tilde{Y}_z^1} Q_z = \tilde{Q}_z^1$, which we can easily show is Δ. For the second equality, it is easy to show that if $\tilde{Q}_z^1$ is nonempty, then $U_{par(z)}$ is nonempty using Proposition EC.4. Conversely, if $U_{par(z)}$ is nonempty, then $\tilde{Y}_z^1 \in \tilde{Q}_z^1 = \Delta$. Thus, we get the second equality from Lemma EC.4. We proceed in an identical way if the overlap is $\tilde{Y}_z^2$.

If $z \in T_2$, the overlap is either $\tilde{Y}_z^3$ or $\tilde{Y}_z^4$. In a similar way to above, it can be shown that

$$proj_{\tilde{Y}_z^3} Q_z^2 = \Delta = proj_{\tilde{Y}_z^3} U_{par(z)}.$$

The conclusion is identical if the overlap is $\tilde{Y}_z^4$. □

Appendix C: Proof of Theorem 2

C.1. Proof of Proposition 2

This proof relies on some concepts, such as *chordal completions*, and the relationship between a tree decomposition and a chordal completion. We define these concepts first, then illustrate them and the proof using Figure EC.1.

Definition EC.2 (Chordal graph.). A graph G is chordal if all cycles of four or more vertices have a chord, that is, an edge that connects two vertices of the cycle without being part of the cycle.

For example, G in Figure EC.1a is chordal as it is complete. However, G' in Figure EC.1b is not chordal: nodes S_1, SI_2, S_2, SI_3 form a cycle with no chord.

Definition EC.3 (Chordal completion). A chordal completion of a graph G is a chordal graph, on the same vertex set, that has G as a subgraph.

Figure EC.1c provides the chordal completion of the graph in Figure EC.1b. The added chords are the orange lines.

Definition EC.4 (Maximal clique). A clique C in a graph G is a set of vertices that are all pairwise adjacent. We say that C is maximal if no other vertex of G is adjacent to all vertices of C.

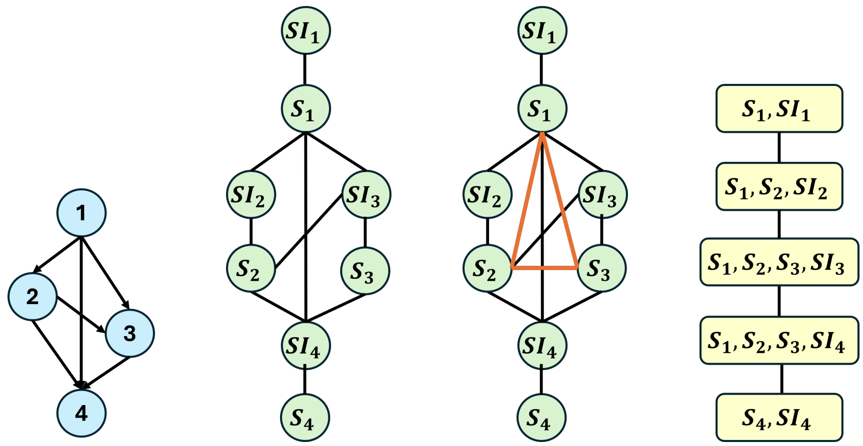
In Figure EC.1c, the triangle S_1, SI_2, S_2 forms a maximal clique, as does S_1, S_2, S_3, SI_3.

Proposition EC.5 **(see, e.g., Sections 1.3 and 2.3 in Kammer 2010)**. *The following results hold:*

(i) The treewidth of a graph G is the minimum size over all chordal completions of G of the largest clique in this completion minus one.

(ii) Given a chordal graph G and maximal cliques in G, there exists a tree decomposition of G such that each and every one of its bags contains exactly the vertices of a maximal clique in G (different to all other maximal cliques contained in other bags). Its width is thus the size of the largest clique in G minus one.

A tree decomposition as in (ii) is also called a *clique tree*. An example is given in Figure EC.1d for the graph in EC.1c.



(a) G (b) G' (c) G'_C (d) $\mathcal{T}'$

Figure EC.1 An example where G is the complete graph over 4 vertices, with corresponding intersection graph G', its chordal completion G'_C, and its tree decomposition $\mathcal{T}'$.

We are now ready to prove Proposition 2.

Proof of Proposition 2. Given a graph G with treewidth ω, build the intersection graph G' for (2) following Proposition 1. We now show that there exists a tree decomposition of G' with treewidth at most ω and with exactly one SI_j in each bag. From Proposition EC.5, as G has treewidth ω, there exists a chordal completion G_C of G giving rise to the treewidth ω (i.e., it is the size of its largest clique minus one). Note that this chordal completion G_C can be taken to be directed and acyclic: consider the initial directed and acyclic graph G. There exists a topological ordering of the nodes from G. Building G_C involves adding edges to G such that each cycle of size larger than or equal to 4 in the graph has a chord: we orient the edges we add using the topological ordering given by G. Thus, G_C has a topological ordering, which implies that it must be directed and acyclic.

We now use G_C to build a chordal completion G'_C of G'. We start by taking G'_C to be G_C where we replace (as in Proposition 1) any node i by the directed graph $SI_i \rightarrow S_i$ and then consider the undirected versions of G'_C and G_C in the remainder. Note that G'_C is bipartite. Thus, it does not have any cycles of uneven size. Furthermore, it does not have any cycle of size 6 or more which are chordless. Indeed, consider a cycle of size 8 or more in G'_C with no chord. Due to the bipartite nature of G'_C, it must be of the type $SI_{i_1} - S_{j_1} - SI_{i_2} - S_{j_2} - SI_{i_3} - S_{j_3} - SI_{i_4} - S_{i_4} - SI_{i_1}$, where j_1, j_2, j_3, j_4 can possibly overlap with i_1, i_2, i_3, i_4, but are distinct from one another. Consider the union of both sets of indexes: it has cardinality of minimum 4. By construction, there must be a cycle in G_C indexed by the union of these two sets, which is of size 4 or more and has no chord: this contradicts the chordal nature of G_C, so there are no chordless cycles of size 8 or more in G'_C. Now consider a cycle of size 6 with no chord in G'_C: it is possible to construct such a cycle involving variables with only three distinct indexes (other 6-cycles would involve at least 4 different indexes for the variables, and we can revert back to the 8-cycle case). This 6-cycle would necessarily be of the type $S_i - SI_j - S_j - SI_k - S_k - SI_i - S_i$ for some nodes i, j, k. This would imply that there are the directed edges (i, j),

(j,k), and (k,i) in G_C (when considered as a directed graph) given the construction of G'_C. However, G_C is taken to be acyclic: thus, such a set-up cannot occur and there are no chordless cycles of size 6 or more in G'_C.

Hence, it only remains to add chords to cycles of size 4 to make G'_C a chordal completion of G'. Chordless cycles of size 4 in G'_C are necessarily of the type $S_i - SI_j - S_j - SI_k$ where i,j,k are nodes in G as cycles of the type $S_i - SI_j - S_j - SI_i$ are once again precluded by G being a directed acyclic graph. We add a chord to cycles of this type by adding an edge between S_i and S_j. This turns G'_C into a chordal completion of G'.

Note that any clique in G'_C cannot contain two variables SI_{j_1} and SI_{j_2} with $j_1 \neq j_2$. This is because G'_C is initially bipartite (thus there are no edges between the variables SI_j) and the only edges added on are between variables S_i and S_j. It follows that no maximal clique contains more than one SI_j. Hence, consider two cases:

If a maximal clique contains no variable SI_j (that is, all variables are $\{S_i\}$), then its size ω' cannot be larger than $\omega+1$, where ω is the treewidth of G. Indeed, suppose $\omega' > \omega+1$. Then, there are ω' variables $\{S_i\}$ with an edge between all pairs. From our construction above, these must come from the presence of cycles of the type $S_i - SI_j - S_j - SI_k$ in G'_C, i.e., there is an edge between i and j in G_C. Thus, there are ω' nodes in G_C which pairwise all share edges: this implies that G_C has a clique of size $\omega' > \omega+1$, which contradicts the fact that the largest clique of G_C has size $\omega+1$.

If a maximal clique contains exactly one variable SI_j, then its size ω' cannot be larger than $\omega+2$. Indeed, suppose $\omega' > \omega+2$. Then, there are $\omega'-1$ variables $\{S_i\}_{i\in I}$ for some set I contained in the maximal clique. A similar reasoning as above gives us that there must be pairwise edges between all $\omega'-1$ variables and a clique in G_C of size $\omega'-1 > \omega+1$, which contradicts the fact that the size of the largest clique in G_C has size $\omega+1$.

From (ii) in Proposition EC.5, there exists a tree decomposition of G' where each bag is a maximal clique in G'_C. In bags containing only $\{S_i\}$ variables, we add a variable SI_k by considering the closest bag (arbitrarily breaking ties if needed) with a variable SI in it, and duplicating that variable in all bags on the path. This maintains the tree decomposition properties without increasing the width of the tree as bags that only contain $\{S_i\}$ variables are of size at most $\omega+1$. The width of such a tree is then $\omega+2-1=\omega+1$ and it contains exactly one variable SI_j per bag. □

The mechanics of this proof are illustrated in Figure EC.1. Here, G and G_C are the same as G is the complete graph on 4 nodes. We build the intersection tree G' in Figure EC.1b: it is not chordal, however, the only chordless cycles it has are of length 4, as shown in the proof. We give the chordal completion of G' in Figure EC.1c, using the approach described in the proof. The corresponding clique tree is given in Figure EC.1d. Note that there is at most one SI_j in each bag and that its width is less than or equal to that of G plus one, i.e., 4.

C.2. A detour via another optimization problem

In the rest of the proof, we do not work directly with (14). Instead, we work with an equivalent linear program, given in (EC.3). This is because it is much easier to show that the feasible set of (EC.3) is integral. Theorem 2 remains true nevertheless, as we show that any integral solution in (EC.3) maps to an integral solution in (14) and vice-versa (see Proposition EC.6). In practice, one would implement (14) as it involves fewer variables and constraints; thus, (EC.3) is purely a theoretical construct for the proof of Theorem 2.

We start by introducing some notation and definitions that are required to define the new linear program. Let $(1,\ldots,T_0)$ be a labeling of the nodes of T as in Lemma EC.6. Recall that given a node $z \in T$, $par(z)$ is the (unique) parent of $z \in T$. For $z = 2,\ldots,T_0$, we will be interested in the variables present in the overlap between the bags X_z and $X_{par(z)}$, i.e., in $X_z \cap X_{par(z)}$. For example, in the running example in Figure 2, we have, e.g., $X_3 \cap X_{par(3)} = X_3 \cap X_2 = \{S_1, S_2\}$ and $X_4 \cap X_{par(4)} = X_4 \cap X_3 = \{S_2, SI_3\}$. We define, for $z \in (2, \dots, T_0)$,

$$\hat{I}_z = \{i \in I_z \mid S_i \in X_z \cap X_{par(z)}\} = I_z \cap I_{par(z)} \text{ and } \hat{J}_z = \{j \in J_z \mid SI_j \in X_z \cap X_{par(z)}\} = J_z \cap J_{par(z)}. \quad \text{(EC.2)}$$

The set $\hat{I}_z$ (resp. $\hat{J}_z$) contains the variables S_i (resp. SI_j) present in the overlap of X_z and $X_{par(z)}$. Once again, using our running example, $\hat{I}_3 = \{1,2\}$ whereas $\hat{I}_4 = \{2\}$ and $\hat{J}_4 = \{3\}$. Note that $\hat{I}_z$ or $\hat{J}_z$ can be empty. If that is the case, then any index which comes from this set or any summation over this set can be dropped.

We are now ready to introduce new variables which track the values taken on by the variables in the overlaps $X_z \cap X_{par(z)}$. Using a similar convention to before, we write: $\left\{w^{\{S_i\}_{i\in\hat{I}_z}\{SI_j\}_{j\in\hat{J}_z}}_{\{k_i\}_{i\in\hat{I}_z}\{\ell_j\}_{j\in\hat{J}_z}}\right\}^z_{\{k_i\}_{i\in\hat{I}_z},\{\ell_j\}_{j\in\hat{J}_z}}$, where k_i corresponds to the value taken on by S_i for $i \in \hat{I}_z$ and ℓ_j corresponds to the value taken on by SI_j for $j \in \hat{J}_z$. We now write the linear program of interest:

$$\min \quad \sum_{j=1}^{n}\sum_{\ell_j=0}^{M}\sum_{k_j=0}^{\Gamma_j} f_j(\ell_j - k_j) r^j_{k_j\ell_j} \quad \text{(EC.3)}$$

$$\text{s.t.} \quad r^j_{k_j\ell_j} \le \left\lfloor \frac{T_j}{k_j - \ell_j} \right\rfloor, \ \forall k_j = 0,\dots,\Gamma_j,\ \ell_j = 0,\dots,M,\ k_j > \ell_j, j = 1,\dots,n \quad (EC.3a_j)$$

$$s^{\{S_i\}_{i\in I_z}S_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}} = 0,\ \forall((k_i)_{i\in I_z},\ell_{j_z}) \in \mathcal{S}_{i_z j_z},\ (i_z, j_z) \in (I_z, J_z) \cap E,\ z \in T \quad (EC.3b^z_{i_z j_z})$$

$$r^{j_z}_{k_{j_z}\ell_{j_z}} = \sum_{\{k_i \mid i \in I_z, i \neq j_z\}} s^{\{S_i\}_{i\in I_z}S_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}},\ \forall k_{j_z} = 0,\dots,\Gamma_{j_z}, \forall \ell_{j_z} = 0,\dots,M,\ \forall z \in T_1, \quad (EC.3c^z)$$

$$\lambda^{j_z}_{\ell_{j_z}} = \sum_{\{k_i \mid i \in I_z\}} s^{\{S_i\}_{i\in I_z}S_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}},\ \forall \ell_{j_z} = 0,\dots,M,\ \forall z \in T, \quad (EC.3d^z)$$

$$\gamma^{i_z}_{k_{i_z}} = \sum_{\{k_i \mid i \in I_z, i \neq i_z\}} \sum_{\ell_{j_z}} s^{\{S_i\}_{i\in I_z}S_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}, \forall k_{i_z} = 0,\dots,\Gamma_{i_z},\ \forall i_z \in I_z,\ \forall z \in T, \quad (EC.3e^z_{i_z})$$

$$\left\{s^{\{S_i\}_{i\in I_z}SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}\right\}^z_{\{k_i\}_{i\in I_z},\ell_{j_z}} \in \Delta,\ \forall z \in T, \quad (EC.3f^z)$$

$$w^{\{S_i\}_{i\in\hat{I}_z}\{SI_j\}_{j\in\hat{J}_z}}_{\{k_i\}_{i\in\hat{I}_z}\{\ell_j\}_{j\in\hat{J}_z}} = \sum_{\{k_i \mid i \in I_z \cap \overline{\hat{I}_z}\}} \sum_{\{\ell_j \mid j \in J_z \cap \overline{\hat{J}_z}\}} s^{\{S_i\}_{i\in I_z}SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}, \forall k_i, i \in \hat{I}_z, \ell_j, j \in \hat{J}_z,\ z \in T \quad (EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}})$$

$$w^{\{S_i\}_{i\in\hat{I}_z}\{SI_j\}_{j\in\hat{J}_z}}_{\{k_i\}_{i\in\hat{I}_z}\{\ell_j\}_{j\in\hat{J}_z}} = \sum_{\{k_i \mid i \in I_{par(z)} \cap \overline{\hat{I}_{par(z)}}\}} \sum_{\{\ell_j \mid j \in J_{par(z)} \cap \overline{\hat{J}_{par(z)}}\}} s^{\{S_i\}_{i\in I_{par(z)}}SI_{j_{par(z)}}}_{\{k_i\}_{i\in I_{par(z)}}\ell_{j_{par(z)}}}, \forall k_i, i \in \hat{I}_z, \ell_j, j \in \hat{J}_z,\ z \in T \quad (EC.3h^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}})$$

with variables $\{\lambda^j_{\ell_j}\}_{\ell_j,j}, \{\gamma^j_{k_j}\}_{k_j,j}, \{r^j_{k_j\ell_j}\}_{k_j,\ell_j,j}, \left\{s^{\{S_i\}_{i\in I_z}SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}\right\}^z_{\{k_i\}_{i\in I_z},\ell_{j_z}}, \left\{w^{\{S_i\}_{i\in\hat{I}_z}\{SI_j\}_{j\in\hat{J}_z}}_{\{k_i\}_{i\in\hat{I}_z}\{\ell_j\}_{j\in\hat{J}_z}}\right\}^z_{\{k_i\}_{i\in\hat{I}_z},\{\ell_j\}_{j\in\hat{J}_z}}$.

The constraints in ($EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) and ($EC.3h^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) reflect the fact that the indicator function of the variables in $X_z \cap X_{par(z)}$ is equal to the sum of the indicator function of the variables in X_z (resp. $X_{par(z)}$) over all variables in X_z (resp. $X_{par(z)}$) that are not in the overlap. As an example, consider node 3 in Figure 2a: node 3 and node 2 overlap on $\hat{I}_3 = \{S_1, S_2\}$. Thus we have:

$$w^{S_1S_2}_{k_1k_2} = \sum_{\ell_3} s^{S_1S_2SI_3}_{k_1k_2\ell_3} = \sum_{\ell_2} s^{S_1S_2SI_2}_{k_1k_2\ell_2},$$

where the first equality corresponds to ($EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) and the second to ($EC.3h^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$).

We first show that (14) and (EC.3) are equivalent, particularly in terms of integral solutions. This will enable us to focus on (EC.3) in the remainder of the section, even though Theorem 2 holds for (14).

PROPOSITION EC.6. *Problems* (14) *and* (EC.3) *are equivalent. Furthermore, any integer solution of* (EC.3) *is an integer solution of* (14) *and conversely.*

Proof. Any feasible solution to (EC.3) can be made into a feasible solution of (14) by simply retaining the values of the variables that appear in (14). As a consequence, if the solution to (EC.3) is integral, so is that of (14). The constraints ($14a_j$)-($14f^z$) are automatically satisfied as they appear as ($EC.3a_j$)-($EC.3f^z$) in (EC.3) and the objective value is the same in both cases.

Now, given a feasible solution to (14), for any $z \in T$, we simply build the variables $\left\{w^{\{S_i\}_{i\in\hat{I}_z}\{SI_j\}_{j\in\hat{J}_z}}_{\{k_i\}_{i\in\hat{I}_z}\{\ell_j\}_{j\in\hat{J}_z}}\right\}_{\{k_i\}_{i\in\hat{I}_z},\{\ell_j\}_{j\in\hat{J}_z}}$, using ($EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$). It is easy to see that such a solution is integral if the solution to (14) is integral. It must be that this solution is feasible. Indeed, ($EC.3a_j$)-($EC.3f^z$) hold because ($14a_j$)-($14f^z$) hold, and ($EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) holds by construction. Thus, if the solution is infeasible, it means that ($EC.3h^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) does not hold for some $z \in T$, some $i \in \hat{I}_z$ or $j \in \hat{J}_z$. Let us assume wlog that it does not hold for some $i_0 \in \hat{I}_z$ and some value $k^0_{i_0}$ of S_{i_0}. The fact that ($EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) holds, yet ($EC.3h^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}}$) does not, implies that:

$$\sum_{\{k_i | i\in I_z\cap\overline{\hat{I}_z}\}}\ \sum_{\{\ell_j | j\in J_z\cap\overline{\hat{J}_z}\}} s^{\{S_i\}_{i\in I_z}SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}} \neq \sum_{\{k_i | i\in I_{par(z)}\cap\overline{\hat{I}_{par(z)}}\}}\ \sum_{\{\ell_j | j\in J_{par(z)}\cap\overline{\hat{J}_{par(z)}}\}} s^{\{S_i\}_{i\in I_{par(z)}}SI_{j_{par(z)}}}_{\{k_i\}_{i\in I_{par(z)}}\ell_{j_{par(z)}}}$$

As the remaining indexes $\{k_i\}$ and $\{\ell_j\}$ correspond to variables in $\hat{I}_z = I_z \cap I_{par(z)}$ and $\hat{J}_z = J_z \cap J_{par(z)}$, the indexes $\{k_i\}_{i\in\hat{I}_z} \neq k^0_{i_0}$ and $\{\ell_j\}_{j\in\hat{J}_z}$ are indexes for both $s^{\{S_i\}_{i\in I_z}SI_{j_z}}_{\{k_i\}_{i\in I_z}\ell_{j_z}}$ and $s^{\{S_i\}_{i\in I_{par(z)}}SI_{j_{par(z)}}}_{\{k_i\}_{i\in I_{par(z)}}\ell_{j_{par(z)}}}$. We sum over these indexes on both sides of the equality. This leads to $\gamma^{i_0}_{k^0_{i_0}} \neq \gamma^{i_0}_{k^0_{i_0}}$ from ($EC.3e^z_{i_z}$), which is a contradiction. Thus it must be the case that such a solution is feasible. It is easy to see that both solutions give rise to the same optimal value. □

Moving forward, we operate with (EC.3) only. The proof follows a very similar outline to that in Appendix B. In particular, we let the feasible set of (EC.3) be P^{ω}_{lin}. Note that we do not need the introduction of an additional optimization problem in the tree case as the overlap between $(X_1 \cup \ldots X_{par(z)}) \cap X_z$ is one set of variables, which is either $\{\lambda^j_{\ell_j}\}$ or $\{\gamma^j_{k_j}\}$. Thus, no additional variables are needed as the role of the variables introduced here are already played by $\{\lambda^j_{\ell_j}\}$ or $\{\gamma^j_{k_j}\}$.

C.3. Proof of Theorem 2

Before proceeding with Steps 2 and 3 of the proof, we need to introduce some new notation, as done in Appendix B. First let $(1, \ldots, T_0)$ be a labeling of the nodes of the tree as given in Lemma EC.6. Recall the definitions of T_1 and T_2 for this section as given in (12). We now introduce Y^1_z and Y^2_z, that is the set of binary variables that are present in each node $z \in T_1$ or $z \in T_2$. We also introduce the associated polytopes Q^1_z and Q^2_z. Finally, we introduce sets of variables and associated polytopes which correspond to overlaps between node z and $par(z)$. This is summarized in Table EC.2, which is the counterpart of Table EC.1.

PROPOSITION EC.7. *We have:*

(i) For any $i \in \{1, \ldots, 3\}$ *and any* $z \in T$*, there exists a point in* $\tilde{R}^i_z$ *if and only if there exists an integral point in* $\tilde{Q}^i_z$.

(ii) For any $z \in T$*, there exists a point in* $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} R^1_z\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} R^2_z\right)$ *if and only if there exists an integral point in* $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} Q^1_z\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} Q^2_z\right)$.

Proof. We show (i) first. If $i=1$, if there exists a point in $\tilde{R}^1_z$, then take $\gamma^i_{k_i} = \mathbf{1}_{S_i=k_i}$ for $i \in \hat{I}_z$ and $w^{\{S_i\}_{i\in\hat{I}_z}}_{\{k_i\}_{i\in\hat{I}_z}} = \mathbf{1}_{\{S_i=k_i\}_{i\in\hat{I}_z}}$. These solutions are integral. It is easy to see that one can find integral $\left\{s^{\{S_i\}_{i\in I_v} SI_{j_v}}_{\{k_i\}_{i\in I_v}\ell_{j_v}}\right\}_{\{k_i\}_{i\in I_v},\ell_{j_v}}$ for $v \in \{z, par(z)\}$ such that $\tilde{Q}^1_z$ is nonempty. Conversely, if $\tilde{Q}^1_z$ is nonempty and integral, then from $(EC.3f^z)$ and $(EC.3g^z_{\{i\}_{i\in\hat{I}_z}\{j\}_{j\in\hat{J}_z}})$, we have that $\left\{w^{\{S_i\}_{i\in\hat{I}_z}}_{\{k_i\}_{i\in\hat{I}_z}}\right\}_{\{k_i\}_{i\in\hat{I}_z}} \in \Delta$ and is binary. Thus, there exist $\{k'_i\}_{i\in\hat{I}_z}$ such that $w^{\{S_i\}_{i\in\hat{I}_z}}_{\{k'_i\}_{i\in\hat{I}_z}} = 1$ with all over values equal to zero. We let $S_i = k'_i$ for $i \in \hat{I}_z$ and one can easily check that such a solution is in $\tilde{R}^1_z$. We proceed similarly for $i=2$: if there exists a point in $\tilde{R}^2_z$, then take $\gamma^i_{k_i} = \mathbf{1}_{S_i=k_i}$, for $i \in \hat{I}_z$, $\lambda_{\ell_{j_z}} = \mathbf{1}_{SI_{j_z}=\ell_{j_z}}$, and $w^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}}_{\{k_i\}_{i\in\hat{I}_z}\ell_{j_z}} = \mathbf{1}_{(S_i=k_i)_{i\in\hat{I}_z} SI_{j_z}=\ell_{j_z}}$, which are integral. From this, one can easily construct a point in $\tilde{Q}^2_z$. The converse is identical to case 1. For $i=3$, if there exists a point in $\tilde{R}^3_z$, then we simply define a new variable compared to case 2, which is $r^{j_z}_{k_{j_z}\ell_{j_z}} = \mathbf{1}_{S_{j_z}=k_{j_z}, SI_{j_z}=\ell_{j_z}}$ which is also integral and one can build a point in $\tilde{Q}^3_z$ from this point. The converse is identical to case 1.

We now show (ii). If $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} R^1_z\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} R^2_z\right)$ is non-empty, then take

$$
\begin{aligned}
&\lambda^{j_{z'}}_{\ell_{j_{z'}}} = \mathbf{1}_{SI_{j_{z'}}=\ell_{j_{z'}}}, \forall \ell_{j_{z'}} = 0,\ldots,T,\ z' \in \{1,\ldots,z\},\\
&\gamma^i_{k_i} = \mathbf{1}_{S_i=k_i}, \forall k_i = 0,\ldots,\Gamma_i, i \in I_{z'},\ z' \in \{1,\ldots,z\},\\
&r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}} = \mathbf{1}_{S_{j_{z'}}=k_{j_{z'}}, SI_{j_{z'}}=\ell_{j_{z'}}},\ \forall k_{j_{z'}} = 0,\ldots,T_{j_{z'}}, \ell_{j_{z'}} = 0,\ldots,M,\ z' \in \{1,\ldots,z\}\cap T_1,\\
&w^{\{S_i\}_{i\in\hat{I}_{z'}},\{SI_j\}_{j\in\hat{J}_{z'}}}_{\{k_i\}_{i\in\hat{I}_{z'}},\{\ell_j\}_{j\in\hat{J}_{z'}}} = \mathbf{1}_{\{S_i=k_i\}_{i\in\hat{I}_{z'}},\{SI_j=\ell_j\}_{j\in\hat{J}_{z'}}},\ \forall \{k_i\}_{i\in\hat{I}_{z'}},\ \{\ell_j\}_{j\in\hat{J}_{z'}},\ z' \in \{2,\ldots,z\}\\
&s^{\{S_i\}_{i\in I_{z'}}, SI_{j_{z'}}}_{\{k_i\}_{i\in I_{z'}},\ell_{j_{z'}}} = \mathbf{1}_{\{S_i=k_i\}_{i\in I_{z'}}, SI_{j_{z'}}=\ell_{j_{z'}}},\ \forall \{k_i\}_{i\in I_{z'}},\ \forall \ell_{j_{z'}}, \forall z' = 1,\ldots,z.
\end{aligned}
$$

These are binary variables and, thus, integers. By construction $(EC.3d^{z'})$ and $(EC.3f^{z'})$ hold for any $z' \in \{1,\ldots,z\}$, as do $(EC.3c^{z'})$ for any $z' \in \{1,\ldots,z\}\cap T_1$, $(EC.3e^{z'}_{i_z})$ for any $i_z \in I_{z'}$ and $z' \in \{1,\ldots,z\}$, and $(EC.3g^{z'}_{\{i\}_{i\in\hat{I}_{z'}}\{j\}_{j\in\hat{J}_{z'}}})$ and $(EC.3h^{z'}_{\{i\}_{i\in\hat{I}_{z'}}\{j\}_{j\in\hat{J}_{z'}}})$ for any $z' \in \{1,\ldots,z\}$. For constraint $(EC.3a_{j_{z'}})$, the argument is identical to the proof of Proposition EC.2 for the tree case and holds for any $z' \in T$. For $(EC.3b^{z'}_{i_{z'}j_{z'}})$, suppose that $(2b_{i_{z'}j_{z'}})$ holds for $(S_{i_{z'}}, SI_{j_{z'}})$. Moreover, suppose that $SI_{j_{z'}} = \ell_{j_{z'}}$ and $S_{i_{z'}} = k_{i_{z'}}$ for some $k_{i_{z'}}, \ell_{j_{z'}}$. As $SI_{j_{z'}} - S_{i_{z'}} \geq 0$, we must have that $\ell_{j_{z'}} \geq k_{i_{z'}}$. Thus, it cannot be that $k_{i_{z'}} > \ell_{j_{z'}}$ and so $s^{\{S_i\}_{i\in I_{z'}} SI_{j_{z'}}}_{\{k_i\}_{i\in I_{z'}}\ell_{j_{z'}}} = 0$ when $k_{i_{z'}} > \ell_{j_{z'}}$. Thus, $(EC.6b^{z'}_{i_{z'}j_{z'}})$ holds given the definition of $\mathcal{S}_{ij}$ in (13). This implies that there exists an integral point in $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} Q^1_z\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} Q^2_z\right)$.

Now suppose that $\left(\wedge_{z'\in\{1,\ldots,z\}\cap T_1} Q^1_z\right) \wedge \left(\wedge_{z'\in\{1,\ldots,z\}\cap T_2} Q^2_z\right)$ contains an integral point. As $Y^1_{z'}$ for $z' \in T_1$ and $Y^2_{z'}$ for $z' \in T_2$ are integral, and given the constraints of $Q^1_{z'}$ and $Q^2_{z'}$, it follows that all variables are binary. Thus, for any $z' \in \{1,\ldots,z\}$, we can define:

$$
S_{i_{z'}} = \sum_{k_{i_{z'}}=0}^{\Gamma_{i_{z'}}} k_{i_{z'}} \cdot \gamma^{i_{z'}}_{k_{i_{z'}}} \quad \text{and } SI_{j_{z'}} = \sum_{\ell_{j_{z'}}=0}^{M} \ell_{j_{z'}} \cdot \lambda^{j_{z'}}_{\ell_{j_{z'}}}.
$$

The fact that $(2c_{j_{z'}})$ and $(2d_{j_{z'}})$ hold is immediate for any $z' \in \{1,\ldots,z\}$. Now, let $z' \in \{1,\ldots,z\}\cap T_1$. For constraint $(2a_{j_{z'}})$, constraints $(EC.3f^{z'})$ and $(EC.3c^{z'})$ imply that $\{r^{j_{z'}}_{k_{j_{z'}}\ell_{j_{z'}}}\}_{k_{j_{z'}},\ell_{j_{z'}}} \in \Delta$, and thus as it is binary, there exist $\hat{\ell}_{j_{z'}} \in \{0,\ldots,M\}$ and $\hat{k}_{j_{z'}} \in \{0,\ldots,\Gamma_{j_{z'}}\}$ such that $r^{j_{z'}}_{\hat{k}_{j_{z'}},\hat{\ell}_{j_{z'}}} = 1$ with $r^{j}_{k_{j_{z'}}\ell_{j_{z'}}} = 0$ for any $(k_{j_{z'}},\ell_{j_{z'}}) \neq (\hat{k}_{j_{z'}},\hat{\ell}_{j_{z'}})$. Coupled with $(EC.3d^{z'})$ and $(EC.3e^{z'}_{i_{z'}})$, we get that

$$
\gamma_{k^{j_{z'}}_{j_{z'}}} = \sum_{\ell_{j_{z'}}} r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}},\ \lambda_{\ell^{j_{z'}}_{j_{z'}}} = \sum_{k_{j_{z'}}} r^{j_{z'}}_{k_{j_{z'}},\ell_{j_{z'}}},
$$

which implies that $S_{j_{z'}} = \hat{k}_{j_{z'}}$ and $SI_{j_{z'}} = \hat{\ell}_{j_{z'}}$. If $\hat{k}_{j_{z'}} \leq \hat{\ell}_{j_{z'}}$, then $S_{j_{z'}} - SI_{j_{z'}} = \hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}} \leq T_{j_{z'}}$ trivially holds as $T_{j_{z'}} \geq 0$. Now suppose that $\hat{k}_{j_{z'}} > \hat{\ell}_{j_{z'}}$. Constraint ($EC.3a_{j_{z'}}$) implies that $r_{\hat{k}_{j_{z'}}\hat{\ell}_{j_{z'}}} \leq T_{j_{z'}}/(\hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}})$, i.e., $(\hat{k}_{j_{z'}} - \hat{\ell}_{j_{z'}}) = S_{j_{z'}} - SI_{j_{z'}} \leq T_{j_{z'}}$, which is ($2a_{j_{z'}}$). Now, let $z' \in \{1,\ldots,z\}$. From ($EC.3d^{z'}$) and ($EC.3e^{z'}_{i_{z'}}$), we have that:

$$SI_{j_{z'}} - S_{i_{z'}} = \sum_{\ell_{j_{z'}}=0}^{M} \ell_{j_{z'}} \cdot \lambda^{j_{z'}}_{\ell_{j_{z'}}} - \sum_{k_{i_{z'}}=0}^{\Gamma_{i_{z'}}} k_{i_{z'}} \cdot \gamma^{i_{z'}}_{k_{i_{z'}}} = \sum_{\ell_{j_{z'}}} \sum_{\{k_i | i \in I_z\}} (\ell_{j_{z'}} - k_{i_{z'}}) s^{\{S_i\}_{i\in I_{z'}}, SI_{j_{z'}}}_{\{k_i\}_{i\in I_{z'}}, \ell_{j_{z'}}}.$$

From the definition of $\mathcal{S}_{i_{z'} j_{z'}}$, constraint ($EC.3b^{z'}_{i_{z'} j_{z'}}$) implies that if $k_{i_{z'}} > \ell_{j_{z'}}$, $s^{\{S_i\}_{i\in I_{z'}}, SI_{j_{z'}}}_{\{k_i\}_{i\in I_{z'}}, \ell_{j_{z'}}} = 0$. Thus, $SI_{j_{z'}} - S_{i_{z'}} \geq 0$, which is ($2b_{i_{z'} j_{z'}}$). □

Proposition EC.8. *Problems (2) and (EC.3) are equivalent provided that the variables appearing in (EC.3) are constrained to be integers.*

This proposition is the counterpart of Proposition EC.3 for the tree case.

Proof. Recall that by definition of the tree decomposition of G', for any $j \in \{1,\ldots,n\}$, there exists a node $z \in T_1$ such that $(S_{j_z}, SI_{j_z}) \in X_z$ and for any $(i,j) \in E$, there exists a node $z \in T_2$ such that $(S_{i_z}, SI_{j_z}) \in X_z$. The proof then follows immediately from Proposition EC.7 (ii) with $z = T_0$, further noting that the objective functions of (2) and (5) are equivalent with the choice of variables made in the proof of the proposition. □

Lemma EC.8. *The polytope Q^1_z is integral for any $z \in T_1$ and the polytope Q^2_z is integral for any $z \in T_2$.*

Proof. Let $z \in T_2$. Recall the definition of Q^2_z given in Table EC.2. If we drop the constraints ($EC.3b^z_{i_z j_z}$) for $(i,j) \in (I_z, J_z) \cap E$, then the resulting polytope is a simplex with binary vertices, following Lemma EC.1. Adding back on the constraints, we obtain a simplex with binary vertices once again from Lemma EC.2. Thus Q^2_z is integral.

Suppose now that $z \in T_1$ and recall the definition of Q^1_z in Table EC.2. Similarly to above, if we drop the constraints ($EC.3a_{j_z}$) and ($EC.3b^z_{i_z j_z}$) for $(i,j) \in (I_z, J_z) \cap E$, we obtain a simplex with binary vertices following Lemma EC.1. Adding back the constraints ($EC.3a_{j_z}$), we obtain once again a simplex with binary vertices, following Lemma EC.3, and this happens once again when we add back the constraints ($EC.3b^z_{i_z j_z}$) following Lemma EC.2. Thus Q^1_z is integral. □

Proposition EC.9. *Let $\tilde{Q}^i_z$ for $i \in \{1,2,3\}$ and Q^i_z for $i \in \{1,2\}$ be the sets defined in Table EC.2. We have:*

(i) For $z \in T_1$, if $\tilde{Q}^i_z$ is nonempty for some $i \in \{1,2,3\}$, then Q^1_z is nonempty.

(ii) For $z \in T_2$, if $\tilde{Q}^i_z$ is nonempty for some $i \in \{1,2\}$ then Q^2_z is nonempty.

(iii) For $z \in T$, if $\tilde{Q}^i_z$ is nonempty for some $i \in \{1,2,3\}$, then $\left(\wedge_{z' \in \{1,\ldots,par(z)\} \cap T_1} Q^1_{z'}\right) \wedge \left(\wedge_{z' \in \{1,\ldots,par(z)\} \cap T_2} Q^2_{z'}\right)$ is nonempty.

Proof. We show instead the following statements:

(i) For $z \in T_1$, if $\tilde{R}^i_z$ is nonempty for some $i \in \{1,2,3\}$, then R^1_z is nonempty.

(ii) For $z \in T_2$, if $\tilde{R}^i_z$ is nonempty for some $i \in \{1,2\}$ then R^2_z is nonempty.

(iii) For $z \in T$, if $\tilde{R}^i_z$ is nonempty for some $i \in \{1,2,3\}$, then $\left(\wedge_{z' \in \{1,\ldots,z\} \cap T_1} R^1_{z'}\right) \wedge \left(\wedge_{z' \in \{1,\ldots,z\} \cap T_2} R^2_{z'}\right)$ is nonempty.

The proposition follows immediately from Proposition EC.7. We only show (i) as (ii) is subsumed by (i) and the proof of (iii) is exactly the same as the one in the proof of Proposition EC.4.

To show (i), assume that $\tilde{R}_z^1$ is nonempty first. Thus, the values of $\{S_i\}_{i\in\hat{I}_z}$ are fixed. Take $S_i = 0$ for all $i \in I_z, i \notin \hat{I}_z$ and let $SI_{j_z} = \max\{\max_{i\in\hat{I}_z, i\neq j_z}\{S_i\}, S_{j_z} - T_{j_z}\}$. By construction, such a point is in R_z^1. Now, assume that $\tilde{R}_z^2$ is nonempty. Take $S_i = 0$ for all $i \in I_z, i \notin \hat{I}_z$. Then, $\{S_i\}_{i\in I_z}$ and $\{SI_{j_z}\}$ as constructed belong to R_z^1. Finally, if $\tilde{R}_z^3$ is nonempty, then again, we let $S_i = 0$ for all $i \in I_z, i \notin \hat{I}_z$ to conclude. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. We show by induction on $z \in (1, \dots, T_0)$ that

$$U_z := \left(\wedge_{z'\in\{1,\dots,z\}\cap T_1} Q_z^1\right) \wedge \left(\wedge_{z'\in\{1,\dots,z\}\cap T_2} Q_z^2\right)$$

is integral. By properties of the tree decomposition of the intersection graph G', it is clear that $U_{T_0} = P_{lin}^{\omega}$. Hence, our induction shows that P_{lin}^{ω} is integral and the theorem follows from Propositions EC.6 and EC.7.

Suppose that $z = 1$. If $1 \in T_1$, then Lemma EC.8 shows that Q_1^1 is integral; if $1 \in T_2$ then Lemma EC.8 shows that Q_1^2 is integral. Thus, U_1 is integral. Now suppose that $U_{par(z)}$ is integral and consider U_z. We have $U_z = U_{par(z)} \wedge Q_z^1$ if $z \in T_1$ and $U_z = U_{par(z)} \wedge Q_z^2$ if $z \in T_2$. Our goal is to use Lemma EC.5 to show that U_z is integral. The induction hypothesis gives us that $U_{par(z)}$ is integral and Q_z^i is integral from Lemma EC.7 for any $i \in \{1, 2\}$. Thus it remains to show that the projection of $U_{par(z)}$ and Q_z^i for $i = \{1, 2\}$ onto their common variables is a common simplex. This is what we show now distinguishing following the cases given in Table EC.2.

The variables appearing in Q_z^i are Y_z^i for $i \in \{1, 2\}$ and the variables appearing in$U_{par(z)}$ are the variables $\left(\cup_{z'\in\{1,\dots,par(z)\}\cap T_1} Y_{z'}^1\right) \cup \left(\cup_{z'\in\{1,\dots,par(z)\}\cap T_2} Y_{z'}^2\right)$. From Lemma EC.6, the common variables between the two sets are then $Y_{par(z)}^i \cap Y_z^j$ for $i, j \in \{1, 2\}$. A simple check reveals that this intersection can only be $\tilde{Y}_z^1, \tilde{Y}_z^2$ (this only occurs if at least one of $par(z), z \in T_2$) and $\tilde{Y}_z^3$ (this only occurs if $par(z), z \in T_1$).

Case 1. Suppose that $z, par(z) \in T$. We have that:

$$proj_{\tilde{Y}_z^1} Q_z^1 = proj_{\tilde{Y}_z^1} Q_z^2 = proj_{\tilde{Y}_z^1} U_{par(z)} = D,$$

where

$$D = \left\{ \tilde{Y}_z^1 \mid \gamma_{k_{i_z}}^{i_z} = \sum_{\{k_i \mid i\neq i_z, i\in\hat{I}_z\}} w_{\{k_i\}_{i\in\hat{I}_z}}^{\{S_i\}_{i\in\hat{I}_z}} \forall k_{i_z}, \forall i_z \in \hat{I}_z, \{w_{\{k_i\}_{i\in\hat{I}_z}}^{\{S_i\}_{i\in\hat{I}_z}}\}_{\{k_i\}_{i\in\hat{I}_z}} \in \Delta \right\},$$

which is a simplex by virtue of Lemma EC.1. To see why these equalities hold, note that $D = \tilde{Q}_z^1$ quite straightforwardly. Then, any feasible point in $\tilde{Q}_z^1$ can be extended to Q_z^i for $i \in \{1, 2\}$ and $U_{par(z)}$ from Proposition EC.9. Furthermore, if $Y_z^1 \in Q_z^1$, then it is easy to see that $\tilde{Y}_z^1 \in D$; likewise, if $Y_z^2 \in Q_z^2$, then $\tilde{Y}_z^1 \in D$. Finally, if $(\cup_{z'\in\{1,\dots,par(z)\}\cap T_1} Y_{z'}^1) \cup (\cup_{z'\in\{1,\dots,par(z)\}\cap T_2} Y_{z'}^2) \in U_{par(z)}$, then $\tilde{Y}_z^1 \in D$. Thus, the equalities are a consequence of Lemma EC.4.

Case 2. Suppose that $z, par(z) \in T$ with at least one of $z, par(z) \in T_2$. We have that:

$$proj_{\tilde{Y}_z^2} Q_z^1 = proj_{\tilde{Y}_z^2} Q_z^2 = proj_{\tilde{Y}_z^2} U_{par(z)} = D,$$

where

$$D = \Big\{ \tilde{Y}_z^2 \mid \gamma_{k_{i_z}}^{i_z} = \sum_{\{k_i \mid i\neq i_z, i\in\hat{I}_z\}} \sum_{\ell_{j_z}} w_{\{k_i\}_{i\in\hat{I}_z}\ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \forall k_{i_z}, \forall i_z \in \hat{I}_z,\ \lambda_{\ell_{j_z}}^{j_z} = \sum_{\{k_i \mid i\in\hat{I}_z\}} w_{\{k_i\}_{i\in\hat{I}_z}\ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \forall \ell_{j_z},$$
$$w_{\{k_i\}_{i\in\hat{I}_z}\ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} = 0,\ \forall k_{i_z} > \ell_{j_z} \text{ s.t. } (i_z, j_z) \in E,\ \{w_{\{k_i\}_{i\in\hat{I}_z}\ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}}\}_{\{k_i\}_{i\in\hat{I}_z}, \ell_{j_z}} \in \Delta \Big\},$$

which is a simplex by virtue of Lemmas EC.1 and EC.2. To see why these equalities hold, note that $D = \tilde{Q}_z^2$ quite straightforwardly. Once again, we can extend any point in $\tilde{Q}_z^2$ to Q_z^i for $i \in \{1,2\}$ and $U_{par(z)}$ from Proposition EC.9. Furthermore, if $Y_z^1 \in Q_z^1$ (resp. $Y_z^2 \in Q_z^2$, $(\cup_{z' \in \{1,\ldots,par(z)\} \cap T_1} Y_{z'}^1) \cup (\cup_{z' \in \{1,\ldots,par(z)\} \cap T_2} Y_{z'}^2) \in U_{par(z)}$) then it is easy to see that $\tilde{Y}_z^2 \in D$. Thus, the equalities are a consequence of Lemma EC.4.

Case 3. Let $z, par(z) \in T_1$. We have that:

$$proj_{\tilde{Y}_z^3} Q_z^1 = proj_{\tilde{Y}_z^3} U_{par(z)} = D,$$

where

$$D = \Big\{ \tilde{Y}_z^3 \mid \gamma_{k_{i_z}}^{i_z} = \sum_{\{k_i \mid i \neq i_z, i \in \hat{I}_z\}} \sum_{\ell_{j_z}} w_{\{k_i\}_{i\in\hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \forall k_{i_z}, \forall i_z \in \hat{I}_z,\ \lambda_{\ell_{j_z}}^{j_z} = \sum_{\{k_i \mid i \in \hat{I}_z\}} w_{\{k_i\}_{i\in\hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \forall \ell_{j_z},$$
$$r_{k_{j_z}\ell_{j_z}}^{j_z} = \sum_{\{k_i \mid i \neq j_z, i \in \hat{I}_z\}} w_{\{k_i\}_{i\in\hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \forall k_{j_z}, \forall \ell_{j_z},\ r_{k_{j_z}\ell_{j_z}}^{j_z} \le \left\lfloor \frac{T_j}{k_{j_z} - \ell_{j_z}} \right\rfloor,\ w_{\{k_i\}_{i\in\hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} = 0,\ \forall k_{i_z} > \ell_{j_z} \text{ s.t. } (i_z, j_z) \in E,$$
$$\{ w_{\{k_i\}_{i\in\hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i\in\hat{I}_z} SI_{j_z}} \}_{\{k_i\}_{i\in\hat{I}_z}, \ell_{j_z}} \in \Delta \Big\},$$

which is a simplex by virtue of Lemmas (EC.1), (EC.3), (EC.2). The equalities hold once again using Proposition EC.9 (by noting that $D = \tilde{Q}_z^3$) and Lemma EC.4. □

References for the E-Companion

Kammer F (2010) *Treelike and chordal graphs: Algorithms and generalizations*. Ph.D. thesis, University of Augsburg.

Kim J, Richard J, Tawarmalani M (2022) A reciprocity between tree ensemble optimization and multilinear optimization. *Optimization Online,* https://optimization-online.org/2022/03/8828.

	Integer Space (SI, S)
Bags in T_1	$X_z^1 = \{\{S_i\}_{i \in I_z}, SI_{j_z}\}, \quad j_z \in I_z$ $R_z^1 = \{X_z^1 \mid (2a_{j_z}), (2b_{ij_z})_{(i,j_z) \in E}, (2c_i)_{i \in I_z}, (2d_{j_z})\}$
Bags in T_2	$X_z^2 = \{\{S_i\}_{i \in I_z}, SI_{j_z}\}, \quad j_z \notin I_z$ $R_z^2 = \{X_z^2 \mid (2b_{ij_z})_{(i,j_z) \in E}, (2c_i)_{i \in I_z}, (2d_{j_z})\}$
Overlaps - Case 1	$\tilde{X}_z^1 = \{\{S_i\}_{i \in \hat{I}_z}\}$ $\tilde{R}_z^1 = \{\tilde{X}_z^1 \mid (2c_i)_{i \in \hat{I}_z}\}$
Overlaps - Case 2 z or $par(z) \in T_2$	$\tilde{X}_z^2 = \{\{S_i\}_{i \in \hat{I}_z}, SI_{j_z}\}, \ j_z \notin \hat{I}_z$ $\tilde{R}_z^2 = \{\tilde{X}_z^2 \mid (2b_{ij_z})_{(i,j_z) \in (\hat{I}_z \times J_z) \cap E}, (2c_i)_{i \in \hat{I}_z}, (2d_{j_z})\}$
Overlaps - Case 3 $z, par(z) \in T_1$	$\tilde{X}_z^3 = \{\{S_i\}_{i \in \hat{I}_z}, SI_{j_z}\}, \ j_z \in \hat{I}_z$ $\tilde{R}_z^3 = \{\tilde{X}_z^3 \mid (2a_{j_z}), (2b_{ij_z})_{(i,j_z) \in (\hat{I}_z \times J_z) \cap E}, (2c_i)_{i \in \hat{I}_z}, (2d_{j_z})\}$

	Binary Space $(\lambda, \gamma, r, w, s)$
Bags in T_1	$Y_z^1 = \left\{ \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \{\gamma_{k_i}^i\}_{\{k_i\}_{i \in I_z}}, \{r_{k_{j_z}\ell_{j_z}}^{j_z}\}_{k_{j_z}, \ell_{j_z}}, \left\{ w_{\{k_i\}_{i \in \hat{I}_v} \{\ell_j\}_{j \in \hat{J}_v}}^{\{S_i\}_{i \in \hat{I}_v} \{SI_j\}_{j \in \hat{J}_v}} \right\}_{\{k_i\}_{i \in \hat{I}_v}, \{\ell_j\}_{j \in \hat{J}_v}}^{v \in \{z, ch(z)\}}, \left\{ s_{\{k_i\}_{i \in I_z} \ell_{j_z}}^{\{S_i\}_{i \in I_z} SI_{j_z}} \right\}_{\{k_i\}_{i \in I_z}, \ell_{j_z}} \right\}$ $Q_z^1 = \Big\{ Y_z^1 \mid (EC.3a_{j_z}), (EC.3b_{ij}^z)_{(i,j) \in (I_z, J_z) \cap E}, (EC.3c^z), (EC.3d^z), (EC.3e_i)_{i \in I_z}, (EC.3f^z),$ $(EC.3g_{\{i\}_{i \in \hat{I}_v} \{j\}_{j \in \hat{J}_v}}^v)_{v \in \{z, ch(z)\}}, (EC.3h_{\{i\}_{i \in \hat{I}_v} \{j\}_{j \in \hat{J}_v}}^v)_{v \in \{z, ch(z)\}} \Big\}$
Bags in T_2	$Y_z^2 = \left\{ \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \{\gamma_{k_i}^i\}_{\{k_i\}_{i \in I_z}}, \left\{ w_{\{k_i\}_{i \in \hat{I}_v} \{\ell_j\}_{j \in \hat{J}_v}}^{\{S_i\}_{i \in \hat{I}_v} \{SI_j\}_{j \in \hat{J}_v}} \right\}_{\{k_i\}_{i \in \hat{I}_v}, \{\ell_j\}_{j \in \hat{J}_v}}^{v \in \{z, ch(z)\}}, \left\{ s_{\{k_i\}_{i \in I_z} \ell_{j_z}}^{\{S_i\}_{i \in I_z} SI_{j_z}} \right\}_{\{k_i\}_{i \in I_z}, \ell_{j_z}} \right\}$ $Q_z^2 = \Big\{ Y_z^2 \mid (EC.3b_{ij}^z)_{(i,j) \in (I_z, J_z) \cap E}, (EC.3d^z), (EC.3e_i)_{i \in I_z}, (EC.3f^z), (EC.3g_{\{i\}_{i \in \hat{I}_v} \{j\}_{j \in \hat{J}_v}}^v)_{v \in \{z, ch(z)\}},$ $(EC.3h_{\{i\}_{i \in \hat{I}_v} \{j\}_{j \in \hat{J}_v}}^v)_{v \in \{z, ch(z)\}} \Big\}$
Overlaps Case 1	$\tilde{Y}_z^1 = \left\{ \{\gamma_{k_i}^i\}_{\{k_i\}_{i \in \hat{I}_z}}, \left\{ w_{\{k_i\}_{i \in \hat{I}_z}}^{\{S_i\}_{i \in \hat{I}_z}} \right\}_{\{k_i\}_{i \in \hat{I}_z}} \right\}$ $\tilde{Q}_z^1 = \Big\{ \tilde{Y}_z^1 \mid (EC.3e_i^z)_{i \in \hat{I}_z}, (EC.3f^v)_{v \in \{z, par(z)\}}, (EC.3g_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z), (EC.3h_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z) \Big\}$
Overlaps Case 2 z or $par(z) \in T_2$	$\tilde{Y}_z^2 = \left\{ \{\gamma_{k_i}^i\}_{\{k_i\}_{i \in \hat{I}_z}}, \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \left\{ w_{\{k_i\}_{i \in \hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i \in \hat{I}_z} SI_{j_z}} \right\}_{\{k_i\}_{i \in \hat{I}_z}, \ell_{j_z}} \right\}$ $\tilde{Q}_z^2 = \Big\{ \tilde{Y}_z^2 \mid (EC.3b_{ij_z}^z)_{(i,j_z) \in E, i \in \hat{I}_z}, (EC.3d^z), (EC.3e_i^z)_{i \in \hat{I}_z}, (EC.3f^v)_{v \in \{z, par(z)\}},$ $(EC.3g_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z), (EC.3h_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z) \Big\}$
Overlaps Case 3 $z, par(z) \in T_1$	$\tilde{Y}_z^3 = \left\{ \{\gamma_{k_i}^i\}_{\{k_i\}_{i \in \hat{I}_z}}, \{\lambda_{\ell_{j_z}}^{j_z}\}_{\ell_{j_z}}, \{r_{k_{j_z}\ell_{j_z}}^{j_z}\}_{k_{j_z}, \ell_{j_z}}, \left\{ w_{\{k_i\}_{i \in \hat{I}_z} \ell_{j_z}}^{\{S_i\}_{i \in \hat{I}_z} SI_{j_z}} \right\}_{\{k_i\}_{i \in \hat{I}_z}, \ell_{j_z}} \right\}$ $\tilde{Q}_z^3 = \Big\{ \tilde{Y}_z^3 \mid (EC.3a_{j_z}), (EC.3b_{ij_z}^z)_{(i,j_z) \in E, i \in \hat{I}_z}, (EC.3c^z), (EC.3d^z), (EC.3e_i^z)_{i \in \hat{I}_z}, (EC.3f^v)_{v \in \{z, par(z)\}},$ $(EC.3g_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z), (EC.3h_{\{i\}_{i \in \hat{I}_z} \{j\}_{j \in \hat{J}_z}}^z) \Big\}$

Table EC.2 Sets of variables and polytopes of interest in the proofs of Theorem 2.